THE FEBRUARY MEETING OF THE AMERICAN MATHEMATICAL SOCIETY.

THE one hundred and eighty-second regular meeting of the Society was held in New York City on Saturday, February 26, 1916. The attendance at the morning and afternoon sessions included the following forty-three members:

Mr. J. W. Alexander, II, Mr. W. E. Anderson, Mr. D. R. Belcher, Dr. A. A. Bennett, Professor Joseph Bowden, Professor E. W. Brown, Professor J. G. Coffin, Professor F. N. Cole, Professor Elizabeth B. Cowley, Professor Louise D. Cummings, Dr. H. B. Curtis, Professor L. P. Eisenhart, Professor H. B. Fine, Dr. C. A. Fischer, Professor T. S. Fiske, Professor W. B. Fite, Professor O. E. Glenn, Dr. T. H. Gronwall, Professor C. C. Grove, Professor H. E. Hawkes, Mr. S. A. Joffe, Professor Edward Kasner, Professor C. J. Keyser, Mr. Harry Langman, Mr. G. H. Light, Mr. P. H. Linehan, Professor E. J. Miles, Mr. G. W. Mullins, Mr. George Paaswell, Dr. P. R. Rider, Mr. J. F. Ritt, Dr. Caroline E. Seely, Dr. H. M. Sheffer, Professor L. P. Siceloff, Professor Mary E. Sinclair, Professor P. F. Smith, Mr. C. E. Van Orstrand, Professor Oswald Veblen, Mr. H. E. Webb, Dr. Mary E. Wells, Professor H. S. White, Mr. J. K. Whittemore, Miss E. C. Williams.

The President of the Society, Professor E. W. Brown, occupied the chair, being relieved by Professors Fine, Fiske, and White. The Council announced the election of the following persons to membership in the Society: Mr. L. E. Armstrong, Stevens Institute of Technology; Professor Grace M. Bareis, Ohio State University; Professor G. A. Chaney, Iowa State College; Mr. J. E. Davis, Pennsylvania State College; G. H. Hardy, M.A., Trinity College, Cambridge, England; Mr. Harry Langman, Metropolitan Life Insurance Company, New York City; Mr. E. D. Meacham, University of Oklahoma; Dr. A. L. Nelson, University of Michigan; Mr. Elmer Schuyler, Bay Ridge High School, Brooklyn, N. Y. Six applications for membership in the Society were received.

The Society has recently taken over the stock of the Chicago Papers and Boston Colloquium Lectures, heretofore in the hands of The Macmillan Company. All publications of the

Society, so far as in stock, are now obtainable directly from the main office. The New Haven Colloquium was published by the Yale University Press, and is sold by them.

The following papers were read at this meeting:

(1) Dr. T. H. GRONWALL: "A functional equation in the kinetic theory of gases (second paper)."

(2) Dr. T. H. GRONWALL: "On the zeros of the functions $P(z)$ and $Q(z)$ associated with the gamma function."

(3) Dr. T. H. GRONWALL: "On the distortion in conformal representation."

(4) Dr. C. A. FISCHER: "Equations involving the derivatives of a function of a surface."

(5) Professor E. W. BROWN: "Note on the problem of three bodies."

(6) Dr. H. BATEMAN: "A certain system of linear partial differential equations."

(7) Dr. H. BATEMAN: "On multiple electromagnetic fields."

(8) Mr. A. R. SCHWEITZER: "On a new representation of a finite group."

(9) Mr. A. R. SCHWEITZER: "Definition of new categories of functional equations."

(10) Professor E. B. WILSON: "Critical speeds for flat disks in a normal wind: theory."

(11) Professor E. B. WILSON: "A mathematical table that contains chiefly zeros."

(12) Professor E. B. WILSON: "Changing surface to volume integrals."

(13) Dr. T. H. GRONWALL: "Elastic stresses in an infinite solid with a spherical cavity."

(14) Dr. T. H. GRONWALL: "On the influence of keyways on the stress distribution in cylindrical shafts."

(15) Professor O. E. GLENN: "The formal modular invariant theory of binary quantics."

(16) Professor O. E. GLENN: "The concomitant system of a conic and a bilinear connex."

(17) Dr. P. R. RIDER: "Trigonometric functions for extremal triangles."

(18) Mr. H. S. VANDIVER: "Symmetric functions of systems of elements in a finite algebra and their connection with Fermat's quotient and Bernoulli's numbers (second paper)."

(19) Mr. S. A. JOFFE: "Calculation of eulerian numbers from central differences of zero."

The papers of Dr. Bateman, Mr. Schweitzer, Mr. Vandiver, and the first and third papers of Professor Wilson were read by title. Abstracts of the papers follow below. The abstracts are numbered to correspond to the titles in the list above.

1. In a previous paper (*Annals of Mathematics*, September, 1915) Dr. Gronwall has shown that, $\varphi(\xi, \eta, \zeta)$ being the logarithm of the function defining the distribution of velocities, the functional equation for φ resulting from the Maxwell-Boltzmann fundamental theorem has as general solution the expression $a + b_1\xi + b_2\eta + b_3\zeta + c(\xi^2 + \eta^2 + \zeta^2)$, under the sole assumption of the continuity of φ for all finite values of the variables. In the present paper, it is shown that this condition may be replaced by the weaker one that φ shall be bounded in an arbitrarily small neighborhood of the origin.

2. Completing a result due to Bourguet, Haskins has recently shown (*Transactions*, 1915) that

$$P(z) = \sum_0^\infty \frac{(-1)^\nu}{\nu!} \frac{1}{z+\nu}$$

has one and only one zero in each of the intervals $-2m - \frac{3}{2} < z < -2m - 1$ and $-2m - 2 < z < -2m - \frac{3}{2}$ $(m = 2, 3, \cdots)$. The proof depends on the Budan-Fourier theorem. In the present paper, Dr. Gronwall shows, by a simpler method, that the real zeros of $P(z)$ lie one in each of the intervals $-2m - 1 - 6/(2m)! \leqq z \leqq -2m - 1 - 1/(2m)!$ and $-2m - 2 + 1/(2m+1)! \leqq z \leqq -2m - 2 + 6/(2m+1)!$, and furthermore that $P(z)$ has exactly four complex zeros, the real parts of which lie between $-\frac{1}{4}\frac{7}{}$ and $-\frac{3}{2}$, and the imaginary parts between $-\frac{1}{2}\frac{1}{}$ and $+\frac{1}{2}\frac{1}{}$. Finally it is shown, by an extremely simple application of Picard's theorem, that the entire function $Q(z) = \Gamma(z) - P(z)$ has an infinity of zeros.*

* It should be noted that Bourguet, *Comptes Rendus*, vol. 96 (1883), pp. 1487–1490, indicated approximations to the real zeros of $P(z)$, which are of essentially the same order as those of Dr. Gronwall. Bourguet also asserted that $P(z)$ can have at most four complex zeros. His proof, however, has been criticized by Nielsen, Handbuch der Theorie der Gammafunktion, p. 36, as inaccurate. It is worth noting, however, that by making use of the known analytic character of $P(z)$, and of the difference equation satisfied by it, Bourguet's proof can be made rigorous, and that then it gives, when taken in connection with Haskins's completion of Bourguet's theorem, the further fact that $P(z)$ has exactly four complex zeros. C. N. HASKINS.

3. In the present paper, Dr. Gronwall gives the exact determination of the bounds in Koebe's distortion theorem, which may now be stated in the following, and final, form:

"When the analytic function

$$w = z + a_2 z^2 + \cdots + a_n z^n + \cdots$$

gives the conformal representation of the circle $|z| < 1$ on a simple region D in the w-plane, then we have for $|z| = r$ and $0 < r < 1$

$$\frac{1-r}{(1+r)^3} < \left|\frac{dw}{dz}\right| < \frac{1+r}{(1-r)^3},$$

$$\frac{r}{(1+r)^2} < |w| < \frac{r}{(1-r)^2},$$

except when $w = z/(1 - e^{\alpha i}z)^2$, in which case the upper and lower bounds are reached for $z = re^{-\alpha i}$ and $z = -re^{-\alpha i}$ respectively. When the region D is convex, we have

$$\frac{1}{(1+r)^2} < \left|\frac{dw}{dz}\right| < \frac{1}{(1-r)^2},$$

$$\frac{r}{1+r} < |w| < \frac{r}{1-r},$$

except for $w = z/(1 - e^{\alpha i}z)$, in which case the upper and lower bounds are reached as before."

It is also shown that the convexity bound, i. e. the upper bound of those values of r for which the map of $|z| < r$ is convex, exceeds $2 - \sqrt{3}$ except when $w = z/(1 - e^{\alpha i}z)^2$, where this value is reached. The above results are then extended in two directions, first by assuming the region D invariant for a rotation of $2\pi/n$ about the origin, and second, by assigning a priori the value of the second coefficient in w.

A detailed abstract will appear in the *Comptes rendus*.

4. The functions considered in Dr. Fischer's paper depend on a surface and also on all of the values taken by a given function at points of the surface. Such a function has two partial fonctionnelle derivatives. The condition of integrability of an equation involving these two derivatives is found, and the characteristics are briefly discussed. Similar equa-

tions, involving functions of lines instead of surfaces, are discussed by Lévy in volume 37 of the *Rendiconti del Circolo Matematico di Palermo.*

5. Professor Brown's note gives a form for the equations of motion in the "restricted" case of the problem of three bodies when the first describes an elliptic orbit about the second, and the third is of zero mass. The curves of zero velocity are obtained and some other consequences deduced.

6. Dr. Bateman's first paper appeared in full in the April BULLETIN.

7. A vector field specified by two vectors H and E may be called a multiple electromagnetic field of rank n when the complex vector $M = H + iE$ satisfies the system of partial differential equations

$$\text{rot}_p M = -\frac{i}{a}\frac{\partial M}{\partial t_p}, \quad \text{div}_p M = 0 \quad (p = 1, 2, \cdots, n).$$

Dr. Bateman shows that each component of M is a right-handed multiple wave function of rank n and obtains various expressions for M by generalizing the solutions of Maxwell's equations which have been given by Hertz, Righi and others. A generalization is also obtained of a theorem due to Appell.

8. In previous communications* Mr. Schweitzer has indicated how any finite group can be represented formally by classes of functional equations derived from the generatrix: $f\{u_1, u_2, \cdots, u_{n+1}\} = \psi(x_1, x_2, \cdots, x_{n+1})$, where $u_i = f_i(t_1^{(i)}, t_2^{(i)}, \cdots, t_n^{(i)}, x_i)$ $(i = 1, 2, \cdots, n+1;\ n = 1, 2, 3, \cdots)$. Perhaps the simplest class of functional equations representing any finite group is the class† $\{E(^{1, 2, \cdots, [n+1]}_{i_1, i_2, \cdots, i_{n+1}})\}$. The author obtains the general solution of this class of equations without referring to any particular substitution, and then proves the following theorem: Given any finite group G, there corresponds to this group a group of functional equations $E(G)$ such that (1) the groups G and $E(G)$ are simply isomorphic, (2) every equation of the group $E(G)$ possesses a solution.

*Cf. BULLETIN, vol. 22, pp. 4, 171.

†For the notation see BULLETIN, l. c., p. 4.

9. The categories of functional equations are defined by Mr. Schweitzer by means of the following generatrices:

$$f(u_1, u_2, \cdots, u_{n+1}) = \psi(x_1, x_2, \cdots, x_{n+1}) + \phi(t_1^{(n+2)}, t_2^{(n+2)}, \cdots, t_n^{(n+2)}),$$

where $u_i = f_i(t_1^{(i)}, t_2^{(i)}, \cdots, t_n^{(i)}, x_i)$ $(i = 1, 2, \cdots, n+1;$ $n = 1, 2, \cdots)$ or analogous generatrices obtained by applying the following processes, either separately or in combination: (1) interchanging one of the u's with the symbol $\psi(x_1, x_2, \cdots, x_{n+1})$; (2) transposing homologously the x's on the left-hand side. From the preceding generatrices are derived functional equations by replacing some, none, or all of the t's with x's and assuming that the remaining t's (if any) with the same subscript are identical. When the function ϕ is identically zero, the functional equations thus defined include equations defined previously by the author. A simple example of one of the new types of functional equations is the relation

$$\lambda f\{f(t_1, t_2, \cdots, t_n, x_1), \cdots, f(t_1, t_2, \cdots, t_n, x_{n+1})\} = \lambda f\{\alpha_1(x_{i_1}), \alpha_2(x_{i_2}), \cdots, \alpha_{n+1}(x_{i_{n+1}})\} + \phi(t_1, t_2, \cdots, t_n),$$

where the i_k range over $1, 2, \cdots, n+1$ and are distinct.

10. In calibrating the new wind tunnel at the Massachusetts Institute of Technology, Lieutenant J. C. Hunsaker, U. S. N., instructor in aeronautics, found a critical velocity for disks (2″ to 6″ in diameter) around 10 to 20 miles per hour. By considering the disks as ellipsoids of revolution and the air as a perfect liquid moving irrotationally, Professor Wilson shows that theoretically such a critical velocity, arising from incipient cavitation, should occur at velocities not much greater than those found experimentally. The sharp edges of the disks and the imperfections of the fluid and of its flow would probably lower the theoretical value found. The paper will appear in connection with Lieutenant Hunsaker's in *Smithsonian Miscellaneous Collections*, volume 62, number 4, article X.

11. Statistical results on the distribution of the different digits in the decimal development of an irrational number or in a tabulation of the values of a function appear not to be

numerous. Professor Wilson sets up an infinite series which defines a function that is perfectly healthy from the analytic viewpoint but has this peculiar statistical malady: The irrational value of the function for any terminating decimal value of the argument has in its infinite decimal development only an infinitesimal proportion of digits other than zero. This problem arose incidentally out of conversations with Professor Josiah Royce on probability and statistics.

12. Professor Wilson's note appeared in full in the April BULLETIN.

13. In the present paper, Dr. Gronwall determines the stress distribution in a solid with a spherical cavity, the dimensions of the solid being very large in comparison with the radius of the sphere, and the stress at a large distance from the cavity being assumed as pure tension or compression of constant magnitude T. It is shown that the maximum tensile stress at the surface of the cavity will be

$$\frac{39\lambda + 54\mu}{18\lambda + 28\mu} T,$$

λ and μ being the constants of elasticity. Thus the maximum stress is roughly twice as large as when there is no cavity, which corrects a statement by Larmor (*Philosophical Magazine*, 1892) that the influence of a cavity on the distribution of tensile stresses is negligible.

14. Dr. Gronwall considers a cylindrical shaft, into which is cut a keyway in the shape of a cylinder intersecting the former orthogonally, the shaft being subjected to torsional stress only. This shape of keyway is common enough in practice. The stress distribution is first determined by the methods of the theory of elasticity, and from this exact result an approximate formula is derived, adapted to practical use, and giving the ratio between the maximum stresses in the keyed and the full portions of the shaft as

$$\frac{2}{1 - 3b^2/a^2},$$

a and b being the radii of shaft and keyway respectively.

15. The first part of Professor Glenn's paper deals with methods of constructing formal invariants and covariants. A theorem is given on the determination of a covariant whose leading coefficient is any assigned seminvariant. Fundamental systems modulo 2 of first degree concomitants are derived for the quartic and the quintic.

In the second part it is shown that the higher forms are reducible in terms of their own first degree concomitants modulo 2. It is proved that if the systems of concomitants modulo 2 of the forms of orders 1, 2, 3 are all finite then the system of any form of order m is finite. It is then shown for the binary cubic modulo 2 that every covariant is quasi-reducible, on multiplication by a power of a first degree invariant, in terms of a set of fourteen concomitants.

16. By a ternary transvection process which Professor Glenn described in a paper communicated in February, 1915, he has derived the fundamental system of simultaneous concomitants of the set consisting of a singly quadric and a doubly linear form. The system furnished initially by the method contains above 100 forms. By various reduction processes he has reduced all but 67 forms. These are given as generalized transvectants and in terms of the symbols.

17. The triangles considered in Dr. Rider's paper are formed by extremal arcs. At one vertex the transversality condition of the calculus of variations is satisfied, thus making the triangles correspond to right-angled triangles in euclidean plane geometry. The trigonometric functions are defined as the limits of the ratios of the lengths of the sides as one side approaches zero, where by length is meant the value of a definite integral taken from one extremity of the arc to the other.

18. The present paper consists mainly of extensions along the lines indicated in Mr. Vandiver's first article under the same title (presented to the Society, April, 1913). A conjugate set in a finite algebra A is defined as a system of elements which are reproduced on multiplication of each element by some element of A. Symmetric functions formed by these sets are studied and for the case where A is represented by the incongruent residues modulo m, a rational integer, the

results obtained involve Fermat's quotient and Bernoulli's numbers.

19. In the *Quarterly Journal*, volume 45 (1914), pages 1–51, Professor J. W. L. Glaisher has calculated the first 27 eulerian numbers from certain recurring formulas and has shown that the method was especially advantageous when "curtate" formulas were employed. Mr. Joffe has verified Professor Glaisher's results and has extended the calculations to five more eulerian numbers by a different method based upon the formula

$$E_n = \sum_{m=0}^{n} (-1)^{m+n} e_{m,n},$$

where $e_{m,n}$ denotes $(1/2^m)\delta^{2m}0^{2n}$, and the quantities $\delta^{2m}0^{2n}$ are "central differences of zero." The successive terms $e_{m,n}$ are computed by a continuous process from the recurring formula

$$e_{m,n} = m[me_{m,n-1} + (m + \overline{m-1})e_{m-1,n-1}],$$

and the final values of E_n are verified by the formula

$$E_n = \sum_{m=1}^{n-1} (-1)^{m+n+1}[(m+1)(m+2) - 1]e_{m,n-1}.$$

F. N. COLE,
Secretary.

ON A CONFIGURATION ON CERTAIN SURFACES.

BY PROFESSOR C. H. SISAM.

(Read before the American Mathematical Society, April 21, 1916.)

THE surfaces here under consideration are rational and are generated by conics. They may be represented birationally on the plane in such a way that, to the plane or hyperplane sections of a given surface of the given type, correspond curves of order n having in common an $(n-2)$-fold point P_0 and Δ simple points $P_1, P_2, \cdots, P_\Delta$. We suppose further that $\Delta = 2k$, so that the surface is of even order, that $n > 3$ and that $k > 2$. For simplicity, we suppose that the fundamental points $P_0, P_1, P_2, \cdots, P_{2k}$ are in generic position.

The generating conics on the surface are determined by the

lines in the parametric plane through P_0. The $2k$ conics which correspond to the lines joining P_0 to $P_1, P_2, \cdots, P_{2k}$ are composite in such a way that the points of one component correspond to the points of the line P_0P_i, those of the other component to the directions through P_i.* We denote the former component right lines by the symbols $1, 2, \cdots, (2k)$, the latter by the symbols $1', 2', \cdots, (2k')$, respectively, so that the lines 1 and 1′, 2 and 2′, etc., constitute a composite generating conic.

The directrix curves of lowest order on the surface are 2^{2k-1} curves C^{n-2} of order $n-2$. These curves are determined as follows:

one, by the directions through P_0,

${}_{2k}C_2$ by right lines through two simple fundamental points,

${}_{2k}C_4$ by conics through P_0 and four simple fundamental points,

. . . . . .

one, by the curve of order k, which has a $(k-1)$-fold point at P_0 and passes through $P_1, P_2, \cdots, P_{2k}$,

wherein ${}_iC_j$ denotes the number of combinations of i things j at a time.

The C^{n-2} which is determined by directions through P_0 intersects the lines $1, 2, \cdots, (2k)$. We shall denote it by the composite symbol $1\,2\,3 \cdots (2k)$. If, in this symbol, we put, in all possible ways, primes on an even number of the component symbols we determine, for each of the remaining C^{n-2}, a symbol which indicates the lines on the surface that are intersected by it. Two such C^{n-2} which have 2σ component symbols unlike intersect in $\sigma - 1$ points as may be seen by counting the intersections, not at fundamental points, of the corresponding curves in the parametric plane. In particular, each C^{n-2} is intersected in $k-1$ points by a single C^{n-2} of the system.

We can choose sets of precisely $2k$ of the curves C^{n-2} in such a way that no two curves of the set intersect. In fact, if we birationally transform the parametric plane so that a given curve $C_1{}^{n-2}$ corresponds to the directions at P_0, then the C^{n-2} which do not intersect $C_1{}^{n-2}$ correspond to right lines through pairs of simple fundamental points. The C^{n-2} which correspond to the right lines joining a given simple fundamental point P_i to the remaining simple fundamental points form, with $C_1{}^{n-2}$, a set of the required type. Conversely,

* Clebsch, *Math. Ann.*, vol. 1, p. 266.

any given set can be represented in this way. Any $2k-1$ of the C^{n-2} of such a set determine a unique right line on the surface which intersects them all. Of the $2k$ lines determined in this way by a set, an odd number have primed symbols. Conversely, any set of $2k$ right lines on the surface, such that an odd number have primed symbols, determines a unique set of $2k$ C^{n-2} which do not intersect. The number of such sets is thus 2^{2k-1}. A given C^{n-2} belongs to $2k$ sets. Two given skew C^{n-2} both belong to two sets. Three skew C^{n-2} define a unique set.

Let S denote a set of the given type. Each curve of the set determines a unique C^{n-2} which intersects it in $k-1$ points and which is seen at once from its symbol to intersect each of the other curves of S in $k-2$ points. The $2k$ C^{n-2} determined in this way by the C^{n-2} of S are skew to each other. Hence they form a set S' of the given type. It follows at once from the method of formation that the correspondence between S and S' is involutorial.

We have supposed $n>3$, $k>2$. In case $k=2$, there exist, in addition to the above sets of skew curves, eight sets of four skew C^{n-2} such that all the C^{n-2} of a set intersect a fixed line on the surface. Each line on the surface determines a set and the sets can be arranged in conjugate pairs.

In case $n=3$, the C^{n-2} are right lines and the above configurations are components, merely, of known configurations of right lines on rational surfaces which have their plane sections of genus unity.

A similar theory can be developed when the number of simple fundamental points is odd, $\Delta = 2k+1$, so that the surface is of odd order, but in this case the correspondence between the sets, as defined above, is not one to one.

URBANA, ILL.,
Febr. ary 21, 1916.

ON SEPARATED SETS.

BY PROFESSOR W. A. WILSON.

(Read before the American Mathematical Society, April 29, 1916.)

IN the March number of the BULLETIN appeared a discussion of the definition of Lebesgue integrals given in Pierpont's Theory of Functions of Real Variables, volume II, by Fréchet and the author. The questions there discussed are much simplified if use is made of the *outer* and *inner associated sets* of a point set, concepts due, I believe, to W. H. Young.

These sets are defined in the text mentioned, but for the sake of convenience I shall give their definitions here. They arise at once from the definitions of upper and lower measure of a point set. Let A be the set under consideration. Let it be enclosed in an enumerable set R of rectangular cells, of which the sum of the areas is finite and may be denoted by $\bar{R}$. The minimum of all the possible values of $\bar{R}$ is called the upper measure of A and denoted by $\overline{\text{Meas}}\ A$. Now if a sequence $\{R_n\}$ of the rectangular sets is so chosen that $\lim_{n=\infty} \bar{R}_n = \overline{\text{Meas}}\ A$, their divisor (or set of points common to all of them) will contain A, and will be a measurable point set of measure equal to $\overline{\text{Meas}}\ A$ by the ordinary laws of measurable sets. Such a set is called an outer associated set of A and may be denoted by A_e. There will be an infinity of such sets corresponding to a single A, but for each A_e, $\overline{\text{Meas}}\ A_e = \overline{\text{Meas}}\ A$ and the sets A_e differ only by a set of measure zero. The inner associated sets are defined as follows. Let A be enclosed in a rectangular cell Q, let $B = Q - A$, and let B_e be an outer associated set of B. Let $A_i = Q - B_e$. Then A_i is contained in A, is measurable and $\text{Meas}\ A_i = \text{Meas}\ Q - \text{Meas}\ B_e = \text{Meas}\ Q - \overline{\text{Meas}}\ B = \underline{\text{Meas}}\ A$, by the definition of lower measure. This set A_i is called an inner associated set of A. Young has also shown that any A_i may be regarded as the union of an enumerable set of complete sets C_n contained in A and such that $\lim_{n=\infty} \text{Meas}\ C_n = \underline{\text{Meas}}\ A$. The importance of these sets is obvious;

their existence makes it possible in questions concerning the upper and lower measures of any set A to replace A by a *measurable* set containing A and of measure equal to $\overline{\text{Meas}}\ A$ or contained in A and of measure equal to $\underline{\text{Meas}}\ A$.

Applying these notions to separated sets, we have at once the result that, *if A and B are separated sets, there exist measurable sets A_1 and B_1 enclosing A and B respectively and such that Meas A_1 = $\overline{\text{Meas}}\ A$ and Meas B_1 = $\overline{\text{Meas}}\ B$; and further that the measure of the divisor of any such pair, A_1 and B_1, is zero.* The first part of the theorem is obvious; we may take A_e for A_1 and B_e for B_1. To prove the second part, let A_2 and B_2 be measurable sets enclosing A and B respectively and such that the measure of their divisor is zero according to the definition of separated sets. Let A_3 be the divisor of A_1 and A_2, and A_4 the remainder of A_1. Now A_3 contains A, since both A_1 and A_2 do. Therefore Meas $A_3 \geqq \overline{\text{Meas}}\ A$ = Meas A_1. Since A_3 is also $\leqq A_1$, Meas A_3 = Meas A_1 and Meas $A_4 = 0$. Similar results hold for B_3 and B_4, defined in like manner. Thus the divisor of A_1 and B_1 is contained in the divisor of A_2 and B_2, save for at most a set of zero measure made up from A_4 and B_4. Therefore the measure of the divisor of A_1 and B_1 is zero.

The theorem questioned by Fréchet is the following: *Let A and B be separated sets and C their union; then $\overline{\text{Meas}}\ C$ = $\overline{\text{Meas}}\ A$ + $\overline{\text{Meas}}\ B$.* The proof can now be given without the use of the ϵ_n-enclosures, which seem to have caused all the trouble. Let A_e, B_e, and C_e be outer associated sets of A, B, and C respectively. Let A_1 be the divisor of A_e and C_e; let B_1 be the divisor of B_e and C_e; and let C_1 be the union of A_1 and B_1. Also let D be the divisor of A_1 and B_1. Then $A \leqq A_1 \leqq A_e$, $B \leqq B_1 \leqq B_e$ and $C \leqq C_1 \leqq C_e$; hence Meas A_1 = $\overline{\text{Meas}}\ A$, Meas B_1 = $\overline{\text{Meas}}\ B$, Meas C_1 = $\overline{\text{Meas}}\ C$ and by the previous paragraph Meas $D = 0$. Therefore

$$\begin{aligned} \overline{\text{Meas}}\ C &= \text{Meas}\ C_1 \\ &= \text{Meas}\ A_1 + \text{Meas}\ B_1 - \text{Meas}\ D \\ &= \overline{\text{Meas}}\ A + \overline{\text{Meas}}\ B - 0, \end{aligned}$$

which was to be proved.

We can also go farther and say that *if the set C is the union*

of A and B, and $\overline{\text{Meas}}\ C = \overline{\text{Meas}}\ A + \overline{\text{Meas}}\ B$, the sets A and B are separated. Proceeding as in the previous theorem,

$$\begin{aligned}\overline{\text{Meas}}\ C &= \text{Meas}\ C_1 \\ &= \text{Meas}\ A_1 + \text{Meas}\ B_1 - \text{Meas}\ D \\ &= \overline{\text{Meas}}\ A + \overline{\text{Meas}}\ B - \text{Meas}\ D.\end{aligned}$$

But since $\overline{\text{Meas}}\ C = \overline{\text{Meas}}\ A + \overline{\text{Meas}}\ B$, Meas $D = 0$. Thus we have A and B enclosed in measurable sets A_1 and B_1 respectively, of which the divisor D has measure zero. This is the requirement for separated sets.

Regarding the example used by Fréchet and questioned by Pierpont, it can be shown that *any separated partition of a measurable set A will be made up of measurable sets only.*

It is sufficient to prove this for the case that the partition consists of two sets only. For, let B be any one of an enumerable set of separated sets making up A and let U be the union of the remainder. It is readily seen from the definition of separated sets that B and U are separated. Hence it is sufficient to prove B and U measurable.

To do this let B_e and U_e be outer associated sets of B and U respectively, and let B_1 be the divisor of A and B_e, and U_1 the divisor of A and U_e. Then by previous results B_1 and U_1 are measurable, Meas $B_1 = \overline{\text{Meas}}\ B$ and Meas $U_1 = \overline{\text{Meas}}\ U$, and the measure or their divisor is zero.

As A is the union of B_1 and U_1, the set B_1 consists of B and certain points of U contained in B_1. But the divisor of B_1 and U_1 is a null set, hence the divisor of B_1 and U is a null set. Therefore those points of B_1 not belonging to B have measure zero and thus B is the difference between the measurable set B_1 and a null set. Hence B is measurable. In like manner U is measurable. But B was any set of those making up A_1 and so the theorem is proved.

This with the previous theorem gives the important result that no measurable set can be made up of an enumerable set of non-measurable point sets and have the additive property, *i. e.*, $\overline{\text{Meas}}\ A = \overline{\text{Meas}}\ A_1 + \overline{\text{Meas}}\ A_2 + \cdots$, preserved.

NEW HAVEN, CONN.

SINGULAR POINTS OF TRANSFORMATIONS AND TWO-PARAMETER FAMILIES OF CURVES.

BY DR. W. V. LOVITT.

1. *Introduction.*

In the *Transactions* for October, 1915, I discussed some singularities of a point transformation in three variables

$$(1) \qquad x = \phi(u, v, w), \quad y = \psi(u, v, w), \quad z = \chi(u, v, w).$$

Let a particular one of the singular points in question be denoted by P, and let S denote the surface through P in the uvw-space defined by setting the jacobian of the transformation equal to zero. The point P and the surface S are transformed by (1) into a point P_1 and surface S_1 in the xyz-space.

In the present paper there is found on the surface $S_1(x, y, z)$ a curve (d_1) which is the envelope of a one-parameter family of curves properly chosen from the two-parameter family (1). We find in the uvw-space that plane of directions which transforms into the direction of the curve (d_1) in the xyz-space.

2. *Initial Assumptions.*

Let us consider a real point transformation of three-space

$$(1) \qquad x = \phi(u, v, w), \quad y = \psi(u, v, w), \quad z = \chi(u, v, w)$$

with determinant

$$J(u, v, w) \equiv \begin{vmatrix} \phi_u & \phi_v & \phi_w \\ \psi_u & \psi_v & \psi_w \\ \chi_u & \chi_v & \chi_w \end{vmatrix}.$$

The functions ϕ, ψ, χ are not necessarily analytic but it will be presupposed that

(a) the functions ϕ, ψ, χ are of class C'''* in a neighborhood of the origin $(u, v, w) = (0, 0, 0)$;

* We shall say that a single-valued function f of u, v, w is of class C''' if $f(u, v, w)$ and its partial derivatives of orders one, two, and three are continuous in a region in which f is defined.

(*b*) the following initial conditions are satisfied:

$$\phi(0, 0, 0) = \psi(0, 0, 0) = \chi(0, 0, 0) = 0;$$

(*c*) $J(0, 0, 0) = 0$;

(*d*) at the origin $(u, v, w) = (0, 0, 0)$ at least one of the determinants of the matrix

$$\left\| \begin{matrix} J_u & J_v & J_w \\ \phi_u & \phi_v & \phi_w \\ \psi_u & \psi_v & \psi_w \\ \chi_u & \chi_v & \chi_w \end{matrix} \right\| \tag{2}$$

is different from zero.

There is no loss of generality in assuming, as indicated in the conditions, that the singular point P is at the origin in the uvw-space, and that the transform of P by (1) is the origin P_1 in the xyz-space. Neither will generality be lost if we assume for convenience that the determinant

$$H_1 \equiv \begin{vmatrix} J_u & J_v & J_w \\ \psi_u & \psi_v & \psi_w \\ \chi_u & \chi_v & \chi_w \end{vmatrix} \tag{3}$$

is that one of the matrix (2) which does not vanish at the origin.

By our assumptions (*b*) and (*c*) the equations

$$J(u, v, w) = 0, \quad y = \psi(u, v, w), \quad z = \chi(u, v, w) \tag{4}$$

have the initial solution $(u, v, w, y, z) = (0, 0, 0, 0, 0)$. The hypothesis (*d*) justifies the assumption that the determinant (3) is different from zero, as we have seen. Hence by the usual theorems of implicit functions there exists a neighborhood $(0, 0, 0, 0, 0)_\epsilon$* in which no two solutions (u, v, w, y, z) of equations (4) have the same projection (y, z), and a neighborhood $(0, 0)_\delta$ of the point $(y, z) = (0, 0)$ in which equations (4) determine u, v, w as functions of class C'' of y and z,

$$u = u(y, z), \quad v = v(y, z), \quad w = w(y, z) \tag{5}$$

* For these theorems see Bliss, Princeton Colloquium Lectures, pp. 8–9. By the notation $(0, 0, 0, 0, 0)_\epsilon$ is meant a neighborhood

$$|u| < \epsilon, \quad |v| < \epsilon, \quad |w| < \epsilon, \quad |y| < \epsilon, \quad |z| < \epsilon$$

of the point $(0, 0, 0, 0, 0)$.

defining values (u, v, w, y, z) in the neighborhood $(0, 0, 0, 0, 0)$. By substituting these results in the third of equations (1), a surface

$$(S_1) \qquad x = X(y, z)$$

is found, which is the transform by (1) of the surface S.

3. *The Envelope Curve* d_1.

We now interpret equations (1) as a two-parameter family of curves with the parameters v, w. Under the assumption (d), the surface S_1 is the envelope of the curves (1).* If a one-parameter family of curves be chosen from the set (1), this family will not in general have an enveloping curve. The condition that a curve

$$(d_1) \quad x = x(\alpha) = X[y(\alpha), z(\alpha)], \quad y = y(\alpha), \quad z = z(\alpha)$$

on the surface S_1 shall be an envelope may be derived as follows.†

If we substitute $y(\alpha)$, $z(\alpha)$ in the functions $v(y, z)$, $w(y, z)$ defined by equations (5) two functions $v(\alpha)$, $w(\alpha)$ are determined and a one-parameter family of curves is defined when $v(\alpha)$, $w(\alpha)$ are substituted in (1). These curves are tangent to the curve (d_1) if y and z are determined as functions of α so that

$$x_\alpha = \phi_y y_\alpha + \phi_z z_\alpha = m\phi_u, \quad y_\alpha = m\psi_u, \quad z_\alpha = m\chi_u,$$

u, v, and w being thought of as functions of y and z. The three determinants of the matrix

$$\begin{Vmatrix} \phi_y y_\alpha + \phi_z z_\alpha & y_\alpha & z_\alpha \\ \phi_u & \psi_u & \chi_u \end{Vmatrix}$$

must therefore be zero, *i. e.*, the three equations

$$(\phi_u - \phi_y \psi_u) y_\alpha - \psi_u \phi_z z_\alpha = 0$$

$$\chi_u \phi_y y_\alpha - (\phi_u - \chi_u \phi_z) z_\alpha = 0,$$

$$\chi_u y_\alpha - \psi_u z_\alpha = 0,$$

* W. V. Lovitt, "A type of singular points for a transformation of three variables," *Transactions*, vol. 16 (1915), p. 377.

† Mason-Bliss, "The properties of curves in space which minimize a definite integral," *Transactions*, vol. 9 (1908), pp. 440–466.

must be satisfied. The coefficients of y_u, z_u in these equations cannot all vanish since at least one of the derivatives ϕ_u, ψ_u, χ_u is different from zero at the point P_1. That any two of the equations are a consequence of the third may be shown by expanding the determinant of any pair of the equations and using the relation

$$\phi_u - x_y\psi_u - x_z\chi_u = 0.$$

The determination of a one-parameter family of curves having an enveloping curve (d_1) is therefore to be effected by solving one of the above equations. It has the form

$$A(y, z)y_u + B(y, z)z_u = 0,$$

when u, v, and w are replaced by their values in terms of y and z from equations (5). Since this differential equation is of the first order there exists one and only one integral curve

$$y = y(\alpha), \quad z = z(\alpha)$$

in the yz-plane, passing through the point $y = z = 0$ for $\alpha = 0$. The equations of the family of curves tangent to (d_1) are found by substituting $y(\alpha)$, $z(\alpha)$ in the expressions for v and w in terms of y and z from equations (5) and then putting the resulting functions $v(\alpha)$, $w(\alpha)$ in equations (1). A family of extremals

$$x = \phi(u, \alpha), \quad y = \psi(u, \alpha), \quad z = \chi(u, \alpha)$$

is thus found, which are tangent to (d_1) when $u = u(\alpha)$. The equation of the envelope (d_1) will then be

$$x = \phi[u(\alpha), \alpha], \quad y = \psi[u(\alpha), \alpha], \quad z = \chi[u(\alpha), \alpha].$$

We have then the following theorem:

THEOREM 1: *Given a family of curves*

$$x = \phi(u, v, w), \quad y = \psi(u, v, w), \quad z = \chi(u, v, w),$$

if on a particular curve C_1 the determinant J vanishes at the point P_1 and one at least of the determinants of the matrix (2) *is different from zero at P_1, then the family of curves has an enveloping surface S_1 which touches C_1 at P_1 and for which P_1 is not a singular point. On the surface S_1 there exists a unique curve (d_1) without singular points, which passes through the point P_1 and envelopes a one-parameter family of curves, con-*

taining the curve C_1, which are the transforms by (1) *of the lines, in the uvw-space, parallel to the u-axis.*

4. *The Plane of Directions which Transforms into the Direction of d_1.*

Urner* has shown that the necessary and sufficient condition that two non-tangent curves through P which have their directions distinct from the critical direction be rendered tangent by the transformation, is that the plane of their tangents at the point contain the line having the critical direction. Furthermore each plane of directions through the critical direction is compressed into a single direction. We ask, what is the plane of directions which is compressed into the direction of the envelope (d_1) at P_1?

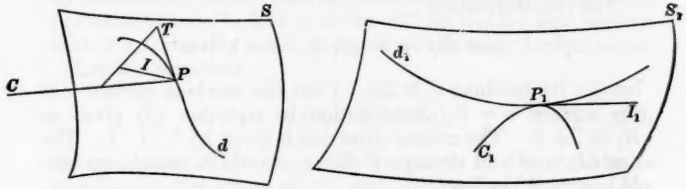
This plane must contain the line of critical direction

$$(I) \qquad \frac{u}{I_1} = \frac{v}{I_2} = \frac{w}{I_3}.$$

It must contain the line

$$(C) \qquad v = w = 0.$$

Thus the plane is completely determined, unless these directions coincide; but this cannot, in general, happen.



Designate by λ the plane determined by C and I. Every direction in λ except I goes into the same direction I_1 in the xyz-space. In particular the direction T which is the intersection of λ with the tangent plane to the surface S at P goes into the direction I_1. To the curve (d_1) on the surface S_1 there corresponds a unique curve (d) on the surface S and from

* S. E. Urner, "Certain singularities of point transformations in space of three dimensions," *Transactions*, vol. 13 (1912), pp. 232–264.

what we have just shown the curve (d) must be tangent to the plane λ at the point P.

We determine the differential equation for the curve (d) as follows: obtain the equation of the plane λ. It is

$$I_3 v - I_2 w = 0.$$

The line T is the intersection of this plane with the tangent plane to the surface S at the point P. The projection of its direction on the vw-plane is

$$I_2 dw - I_3 dv = 0.$$

The integral curve of this equation which passes through the point P is the projection of the curve (d) upon the vw-plane. This integral together with the surface S completely determines the curve (d).

We have then the following theorem:

THEOREM 2. *The plane of directions in the uvw-space determined by the critical direction* I $(I_1 : I_2 : I_3)$ *and the line* $v = w = 0$ *and containing the tangent line* T *to the curve* (d) *transforms into a single direction* I_1 *in the xyz-space. The direction* I_1 *is the direction of the tangent to the curve* (d_1), *which curve is the transform of the curve* (d) *by means of equations* (1).

5. *Illustration.*

The transformation

$$x = u^2, \quad y = u + v, \quad z = v + w$$

has for its jacobian $J \equiv 2u$. Thus the jacobian surface S is the surface $u = 0$. Substitution in equation (3) gives us $H_1 \equiv 2 \neq 0$. The critical direction is given by $1 : 1 : 1$. The line (d) on S and through P $(0, v_0, w_0)$ which transforms into (d_1) on S_1 is the line

$$(d) \qquad v + w = v_0 + w_0, \quad u = 0.$$

The transform of (d) is the line

$$(d_1) \qquad z = v_0 + w_0, \quad x = 0.$$

The surface S_1 is given by $x = 0$. The lines

$$v = c_1, \quad w = c_2, \quad u = t,$$

which are parallel to the u-axis, are transformed into

$$x = (y - c_1)^2, \quad z = c_1 + c_2,$$

which are seen to be tangent to the surface S_1. It is evident from the last equations that those one-parameter families of parabolas which lie in the planes parallel to the xy-plane have envelopes, and that no others have. These envelopes are the curves (d_1).

PURDUE UNIVERSITY.

AN ELEMENTARY BOUNDARY VALUE PROBLEM.

BY PROFESSOR DUNHAM JACKSON.

(Read before the American Mathematical Society, April 29, 1916.)

IT is intuitively obvious that if a simple continuous curve is given in the (x, y)-plane, and a continuous distribution of values along the curve, there will exist functions of x and y which are continuous in both variables together, and which take on the prescribed values along the curve. It is the purpose of the present note to give an analytic proof of this fact, by elementary means, and, in particular, without reference to potential theory.* The problem will be treated first for the case of a rectifiable curve, then for an arbitrary Jordan curve.

Let the equations

$$x = f(s), \quad y = \varphi(s), \qquad (0 \leqq s \leqq l),$$

define a simple closed rectifiable curve C, the variable s standing for the length of arc, and l for the total length of the curve. It is assumed that the functions $f(s)$ and $\varphi(s)$ are continuous throughout their interval of definition, and that $f(0) = f(l)$, $\varphi(0) = \varphi(l)$, but that with this exception no one pair of values (x, y) is given by two distinct values of s. Let $F(s)$ be an arbitrary continuous function defined throughout the same interval, subject to the condition that $F(0) = F(l)$.

* I understand that Mr. R. E. Gleason has had occasion to deal with a similar problem in connection with a paper recently presented to the Society; see BULLETIN, vol. 22 (1916), pp. 278–279.

We shall consider the function $J(x, y) = J_1/J_0$, where

$$J_1(x, y) = \int_0^l \frac{F(s)}{\rho^2} ds, \quad J_0(x, y) = \int_0^l \frac{ds}{\rho^2},$$

and

$$\rho = \sqrt{[x - f(s)]^2 + [y - \varphi(s)]^2}.$$

The function J is obviously defined and continuous throughout the (x, y)-plane, with the exception of the points of the curve C. We shall show that if the point (x, y) approaches a point P_0 of C, with the coordinates $(f(s_0), \varphi(s_0))$, the value of $J(x, y)$ will approach $F(s_0)$ as a limit.* There is clearly no loss of generality in assuming that s_0 is distinct from 0 and l.

We have to deal with the difference

$$J(x, y) - F(s_0) = \frac{1}{J_0} \int_0^l [F(s) - F(s_0)] \frac{ds}{\rho^2}.$$

Let ϵ be any positive quantity. Let $\delta > 0$ be chosen so that $| F(s) - F(s_0) | \leqq \frac{1}{2}\epsilon$ for $| s - s_0 | \leqq \delta$. Then

$$\left| \int_{s_0-\delta}^{s_0+\delta} [F(s) - F(s_0)] \frac{ds}{\rho^2} \right| \leqq \tfrac{1}{2}\epsilon J_0,$$

regardless of the position of the point (x, y), provided only that it does not lie on C. Let γ be the minimum distance from P_0 to a point of C for which $| s - s_0 | \geqq \delta$. This minimum will be positive, since the distance is a continuous function of s which does not reduce to zero. Denoting by ρ_0 the distance

$$\rho_0 = \sqrt{[x - f(s_0)]^2 + [y - \varphi(s_0)]^2},$$

a quantity which depends on x and y but not on s, we can be sure that if $\rho_0 \leqq \frac{1}{2}\gamma$, then $\rho \geqq \frac{1}{2}\gamma$ for $| s - s_0 | \geqq \delta$. Consequently, if M is the maximum of $| F(s) |$,

$$\left| \int_0^{s_0-\delta} [F(s) - F(s_0)] \frac{ds}{\rho^2} \right| \leqq \int_0^{s_0-\delta} 2M \cdot \frac{4ds}{\gamma^2} = \frac{8M(s_0 - \delta)}{\gamma^2};$$

* It is our purpose merely to indicate a single solution of the boundary value problem; when on solution is given, it is of course possible immediately to find infinitely many others.

and similarly

$$\left| \int_{s_0+\delta}^{l} [F(s) - F(s_0)] \frac{ds}{\rho^2} \right| \leqq \frac{8M(l - s_0 - \delta)}{\gamma^2}.$$

We see accordingly that

$$| J(x, y) - F(s_0) | \leqq \tfrac{1}{2}\epsilon + \frac{8Ml}{\gamma^2 J_0}, \tag{1}$$

provided that (x, y) is a point at a distance from P_0 not greater than $\frac{1}{2}\gamma$, and not lying on C. It remains only to show that as (x, y) approaches P_0, the value of J_0 becomes infinite. For if this is established, the second term on the right-hand side of (1) will be less than $\frac{1}{2}\epsilon$ when (x, y) is sufficiently near to P_0, and it will follow that

$$| J(x, y) - F(s_0) | < \epsilon.$$

Let η be an arbitrarily small positive quantity. If $\rho_0 \leqq \frac{1}{2}\eta$ and $| s - s_0 | \leqq \frac{1}{2}\eta$, it is certain that $\rho \leqq \eta$, since it has been assumed that s represents the length of arc along the curve. For all points within a distance $\frac{1}{2}\eta$ of P_0, therefore,

$$J_0 > \int_{s_0-\frac{1}{2}\eta}^{s_0+\frac{1}{2}\eta} \frac{ds}{\rho^2} \geqq \int_{s_0-\frac{1}{2}\eta}^{s_0+\frac{1}{2}\eta} \frac{ds}{\eta^2} = \frac{1}{\eta},$$

which can be made arbitrarily large by taking η sufficiently small. This completes the proof.

Now let C be an arbitrary closed Jordan curve, given by a pair of equations

$$x = f(t), \quad y = \varphi(t), \qquad 0 \leqq t \leqq a,$$

where the functions f and φ are continuous, and do not yield any one point twice, except for $t = 0$ and $t = a$. It will be convenient to think of f and φ as defined for all real values of t, with the period a; they will be continuous without exception. Let $F(t)$ be an arbitrary continuous function of period a.

Let $\omega_1(\eta)$ and $\omega_2(\eta)$ be the maxima of $| f(t'') - f(t') |$ and $| \varphi(t'') - \varphi(t') |$ respectively for $| t'' - t' | \leqq \eta$, and let $\omega(\eta) = \omega_1(\eta) + \omega_2(\eta)$. Then $\omega(\eta)$, defined for $\eta \geqq 0$, is a function which is positive or zero, and never decreases when η increases. Furthermore, $\lim_{\eta=0} \omega(\eta) = 0$, because of the uniform continuity of f and φ; and, more generally, $\omega(\eta)$ is continuous for all positive values of η, since $\omega(\eta' + \eta'')$

$\leqq \omega(\eta') + \omega(\eta'')$. Let $\beta = \chi(\eta) = \omega(\eta) + \eta$. This new function is continuous, reduces to zero for $\eta = 0$, and *always increases* when η increases. To each value of $\beta \geqq 0$ corresponds one and just one value of η; the inverse function

$$\eta = \psi(\beta)$$

is itself increasing and continuous, and, in particular, $\lim_{\beta=0} \psi(\beta) = 0$.

We are ready now to write down a solution of the boundary value problem. We shall set $J(x, y) = J_1/J_0$, where

$$J_1(x, y) = \int_0^a \frac{F(t)dt}{[\psi(\frac{1}{2}\rho)]^2}, \quad J_0(x, y) = \int_0^a \frac{dt}{[\psi(\frac{1}{2}\rho)]^2},$$

and

$$\rho = \sqrt{[x - f(t)]^2 + [y - \varphi(t)]^2}.$$

It is true again that $J(x, y)$ is defined and continuous at all points (x, y) not lying on the given curve. Guided by the earlier demonstration, we shall begin the proof that $J(x, y)$ is a function having the desired property, by showing that J_0 becomes infinite as (x, y) approaches a point P_0: $(f(t_0), \varphi(t_0))$ of the curve.

Let β be an arbitrarily small positive quantity, and $\eta = \psi(\beta)$. If $| t - t_0 | \leqq \eta$, then

$$| f(t) - f(t_0) | \leqq \omega_1(\eta), \quad | \varphi(t) - \varphi(t_0) | \leqq \omega_2(\eta),$$

and hence the distance from P_0 to the point $(f(t), \varphi(t))$ is subject to the inequality

$$\sqrt{[f(t) - f(t_0)]^2 + [\varphi(t) - \varphi(t_0)]^2} \leqq \omega(\eta) \leqq \chi(\eta) = \beta.$$

Denoting by ρ_0 the distance from P_0 to the point (x, y), we see that when

$$\rho_0 < \beta \tag{2}$$

we can be sure that $\rho < 2\beta$ for values of t in the interval just named, and hence

$$\tfrac{1}{2}\rho \leqq \beta, \quad \psi(\tfrac{1}{2}\rho) \leqq \psi(\beta) = \eta.$$

It follows that if (x, y) is within the neighborhood of P_0 defined by the inequality (2),

$$J_0(x, y) > \int_{t_0-\eta}^{t_0+\eta} \frac{dt}{[\psi(\frac{1}{2}\rho)]^2} \geqq \int_{t_0-\eta}^{t_0+\eta} \frac{dt}{\eta^2} = \frac{2}{\eta} = \frac{2}{\psi(\beta)}.$$

As $\psi(\beta)$ approaches zero with β, the assertion with regard to J_0 is justified.

Returning to the consideration of the numerator J_1 and the quotient J, let ϵ be an arbitrarily small positive quantity, and δ a positive quantity such that $|F(t) - F(t_0)| \leqq \frac{1}{2}\epsilon$ for $|t - t_0| \leqq \delta$; let γ be the minimum distance from P_0 to a point of C for which $|t - t_0| \geqq \delta$, a distance which is surely positive,* and finally let M be the maximum value of $|F(t)|$. If $\rho_0 \leqq \frac{1}{2}\gamma$, it is certain that $\rho \geqq \frac{1}{2}\gamma$ for $|t - t_0| \geqq \delta$. We see that the following inequalities hold:

$$\left|\int_{t_0-\delta}^{t_0+\delta} \frac{F(t) - F(t_0)}{[\psi(\frac{1}{2}\rho)]^2}\,dt\right| \leqq \frac{\epsilon}{2}\int_{t_0-\delta}^{t_0+\delta} \frac{dt}{[\psi(\frac{1}{2}\rho)]^2} < \tfrac{1}{2}\epsilon J_0;$$

$$\left|\int_{0}^{t_0-\delta} \frac{F(t) - F(t_0)}{[\psi(\frac{1}{2}\rho)]^2}\,dt\right| \leqq \int_{0}^{t_0-\delta} \frac{2M dt}{[\psi(\frac{1}{4}\gamma)]^2} = \frac{2M}{[\psi(\frac{1}{4}\gamma)]^2}(t_0 - \delta);$$

$$\left|\int_{t_0+\delta}^{a} \frac{F(t) - F(t_0)}{[\psi(\frac{1}{2}\rho)]^2}\,dt\right| \leqq \frac{2M}{[\psi(\frac{1}{4}\gamma)]^2}(a - t_0 - \delta);$$

and by combination of these,

$$|J(x, y) - F(t_0)| = \left|\frac{1}{J_0}\int_0^a \frac{F(t) - F(t_0)}{[\psi(\frac{1}{2}\rho)]^2}\,dt\right| \leqq \frac{\epsilon}{2} + \frac{2Ma}{[\psi(\frac{1}{4}\gamma)]^2 J_0}.$$

If (x, y) is sufficiently near to P_0, the value of J_0 will be so large that the second term on the right is less than $\frac{1}{2}\epsilon$, and we shall have the inequality which establishes the theorem to be proved,

$$|J(x, y) - F(t_0)| < \epsilon.$$

It may be remarked that similar reasoning can be applied to Jordan curves that are not closed, or to a system of any finite number of Jordan curves, no two of which have a point in common.

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* We are assuming here that t is restricted to the interval $0 \leqq t \leqq a$, and are making for convenience the further assumption, of no essential significance, that t_0 is an interior point of the same interval.

CONCERNING REVIEWS.

On reading certain book reviews that have appeared in recent numbers of the Bulletin, one is reminded of Addison's complaint that rather than to dwell upon the excellencies of a work some reviewers imagine they have discharged their duty when they have succeeded in pointing out slight faults and errors, forgetting that

> "Errors, like straws, upon the surface flow;
> He who would search for pearls must dive below."

As an instance justifying this complaint I wish to cite the review of the "Memorabilia Mathematica" in the January number of the Bulletin. Except for an extract from the preface of the book and its table of contents, which is erroneously quoted (there are twenty-one chapter headings instead of the seventeen quoted by the reviewer, and two of those are wrongly quoted), one seeks in vain for a word that would enlighten the reader as to the contents of the book. The remainder of the review, as far as it deals with the work under consideration, is limited to trivial errors and petty fault-finding. Of what possible interest can it be to the reader to be told that in one place T' should be replaced by T', or that "inapt" appears where the original has the misspelled form "unapt," or again, that Newton's utterance "I don't know what I may seem to the world" as quoted by Parton is given by Brewster in the form "I do not know what I may appear to the world"? Surely an author might deem himself fortunate whose work were blemished by no greater faults!

Suppose the reviewer's five-page review were reviewed according to his own standard. He writes "Porton" for "Parton," "Euclyde" for the "Euclide" of the original, and "hut" for "hyt" in the line "Yn Egypte he tawghte hyt ful wyde." His quotation from Prior

> "Circles to square, and cubes to double,
> Would give a man excessive trouble;"

should read

> "Circles to square, and Cubes to double,
> Would give a Man excessive Trouble:"*

* Matthew Prior, Cambridge English Classics, Cambridge (1905), p. 248.

The original form of the line

"God said, 'Let Newton be!' and all was light,"

from Pope's Epitaph intended for Sir Isaac Newton is not

"God said, *Let Newton be*! and all was light,"

as the reviewer has it, but

"GOD said, *Let Newton be*! and all was Light."*

There are other errors of a like character but enough have been cited to lend support to Addison's dictum that there never was a critic who made it his business to lash the faults of others who was not guilty of greater faults himself.

In conclusion I must call attention to one or two more serious errors in the review in question. On page 188 we are told that in Ahrens's Scherz und Ernst in der Mathematik names of living mathematicians are rarely met with. The volume in question contains by actual count 20 quotations from Klein, 18 from Poincaré, 10 from M. Cantor, 7 each from Hilbert and Frobenius, 6 from G. Cantor, and so on through more than a score of names of men either now living or deceased since the book appeared in 1904. Again, the Memorabilia Mathematica contains 1,140 quotations instead of some 1,200 as stated by the reviewer. The seven-line reference on page 190 to the reviewer's own paper is irrelevant to the matter in hand. A most curious slip occurs on page 191 in the reviewer's observation "for Reid, M." read "Reid, T." the line criticized being, "Reid, M. as an exercise in language."

ROBERT E. MORITZ.

THE UNIVERSITY OF WASHINGTON.

WITH regard to the collection of quotations which Professor Moritz edits, the reviewer does not find that he has made a single statement which may be legitimately termed inaccurate, or which is liable to give a wrong impression as to the merits of the editor's redaction—a redaction which the editor himself appears to regard as containing "pearls" unnoticed by the reviewer. Let us see what his strictures amount to.

* Warburton's "The Works of Alexander Pope, as they were delivered to the Editor a little while before his death, etc." London (1760), vol. 6, p. 99.

In spite of the authority of the great Oxford Dictionary he contends that "some 1200" may not be used to refer to 1140! Again, the reviewer is accused of error in tabulation of the contents. These consist, presumably, (1) in stating that the quotations were classed under 20 headings when the editor claims 21 ("Persons and anecdotes, A–M," "Persons and anecdotes, N–Z," the reviewer was guilty of combining under one heading: "Persons and anecdotes"); (2) in asserting that "Definition and object of mathematics" was a heading when the first word should have been in the plural form; and (3) in leaving it to the reader to infer that two of the 20 headings, "Mathematics as a Fine Art" and "Mathematics as a Language," were indicated by "Mathematics as a fine art, as a language"; and similarly that three headings were indicated by "Mathematics and logic, and philosophy, and science."

In response to the inquiries of Professor Moritz the following replies may be vouchsafed: (*a*) In the Thomson and Tait quotation T' has no meaning, while the calculus notation T' is peculiarly pregnant with suggestion; (*b*) it is not true that unapt is misspelled for inapt in Orr's mnemonic—this may be verified by the simple expedient of consulting the Century Dictionary; (*c*) it may be learned from any first-class librarian that Parton is a worthless authority in connection with any statement concerning Newton.

Let us now take up five examples, somewhat different in character, to illustrate Professor Moritz's methods of criticism.

1. In his book he gives two of the seven lines written by Pope as an epitaph on Sir Isaac Newton and refers to those two lines as the epitaph in question; the reviewer remarked that this statement was inaccurate and gave the full quotation, with a reference to Elwin and Courthope's standard edition of Pope's works. In this quotation there is not the slightest misprint in capitalization, in italics, or in punctuation. That some other edition, no more authoritative, has a different capitalization in one line is entirely irrelevant.

2. The same method is applied to the quotation by Prior, again given with exact reference by the reviewer. There is not a particle of variation between the original and that indicated in the review.

3. Again, the reviewer wrote: "on page 405 of the index for Reid, M. read Reid, T." To be more explicit, 8 quotations are attributed to Thomas Reid in the Memorabilia. These

are all incorrectly listed in the index on page 405, under Reid, M. Instead, therefore, of a "most curious slip" on the part of the reviewer, yet another has been made by Professor Moritz himself. But his slips in this sentence are not confined to one, or even two. Without any foundation whatever the reviewer is accused of criticizing a line "Reid, M. as an exercise in language." This line occurs nowhere in the book. True one does find "Reidt, M. as an exercise in language," to which the reviewer made no reference, but here he now finds another slip, for instead of M. should be F.*

4. The reviewer repudiates Professor Moritz's statement of what he wrote concerning Ahrens's work. What he *did* write was as follows: "A 24-page detailed index of subjects and authors provides the means for rapid orientation. Names of living mathematicians are rarely met with, but references to the "old masters" such as Abel, Euclid, Gauss, Helmholtz, Lagrange, Laplace, Steiner and Weierstrass are very numerous." Even if "a score of names" of living mathematicians *may* be found in the 24-page index, the statement of the reviewer has not been shown to be in the smallest degree inaccurate.

5. With exact reference to Halliwell's Rara Mathematica the reviewer quoted some lines referring to Euclid. It is not true that "Euclide" should replace "Euclyde" in that quotation; it is true that "hyt" should replace "hut" in the third line, and the reviewer is glad to have his attention drawn to this slip in proof-reading.

The relation between the reviewer's and critic's statements thus set forth, render impotent the critic's remark concerning "other errors of a like character." In conclusion it may now be added that in his review the reviewer mentioned only a few of the three score of slips which he had noticed in the Memorabilia.

R. C. ARCHIBALD.

* On page 408 the biographer of Lord Kelvin is referred to as Sylvanus (instead of Silvanus) Thompson. The reviewer is indebted to Mr. W. J. Greenstreet for calling his attention to this same slip in his review.

SHORTER NOTICES.

Robert of Chester's Latin Translation of the Algebra of Al-Khowarizmi, with an introduction, critical notes, and an English version. By LOUIS CHARLES KARPINSKI. University of Michigan Studies, Humanistic Series, Vol. XI. New York, The Macmillan Company, 1915. viii + 164 pp. Price $2.

OF the various kinds of contributions to the history of mathematics, those which are based upon a study of original sources are, of course, the most important. It was through contributions of this type that Boncompagni's *Bullettino* exercised its great influence; it is through these studies that the Abhandlungen zur Geschichte der Mathematik have proved so valuable; and it is through its articles based upon original sources that the *Bibliotheca Mathematica* has made its reputation. In this critical study of original sources, however, our own scholars have thus far been very backward. This is entirely natural, because it is first necessary to study the secondary sources in any new branch, and it takes time to discover a problem and to find the opportunity for assisting in its solution. On this account those to whom the story of mathematics has a charm have thus far, in our country, been chiefly occupied in learning the literature and in delving into the pages of Cantor and his predecessors, or in studying as time allowed such works as those of Heath or Braunmühl.

It is for this reason that the appearance of this work by Professor Karpinski is particularly noteworthy. Many scholars publish the results of their studies in one line or another from time to time, and these results are often noteworthy; but to few is it given, as it has been in this case, to suggest a new excursion into one of the by-paths of the academic grove of a country. And yet this is what has been done in the work under review, for it stands as the first noteworthy original study of a European version of an early classic in mathematics.

It is well known that Al-Khowarizmi, early in the ninth century, wrote a work bearing the title "algebr w'almuqabala." It is also well known that this work was translated into Latin by Robertus Retinensis (Ketenensis, de Ketene, Ostiensis, Astensis, or Cestrensis), commonly known today as Robert

of Chester, one of the group of English scholars that revealed the most important scientific works of the Arabs to western Europe in the twelfth century. Here, however, the story comes to an end for most students. Occasionally some reader, browsing in the appendix to Libri's Histoire, comes across the Latin version ascribed to Gherardo of Cremona, and the Rosen translation (1831) from an Arabic manuscript in the Bodleian Library is accessible in any large scientific collection; but Robert of Chester's version has thus far remained beyond the reach of most students. Curiously enough, however, the work was at one time prepared for the printer, by Johann Scheybl, professor of mathematics at Tübingen, probably soon after 1550. For some reason Scheybl's text was never published, and this manuscript was not preserved, like various others of this writer, in the library of his university. After various wanderings it appeared in the stock of a German book dealer about fifteen years ago, and the writer of this review purchased it for a small sum, as an anonymous manuscript on mathematics, for the library of Columbia University. About a year later, upon examining it with some care, the name of Scheybl suggested that it might be one of his manuscripts, and photographs were made of other manuscripts in his hand at Tübingen. A comparison of the handwriting showed at once, and without question, that the Columbia manuscript was a lost one of Scheybl's. Moreover, it was seen that it contained a Latin version of Al-Khowarizmi which differed in various details from the one published by Libri and from the Arabic copy used by Rosen. It is this manuscript which Professor Karpinski has transcribed, translated, and annotated.

The form of the publication is very satisfactory. On the left-hand page is the Latin text; on the right-hand page is the translation; at the foot of each page are such notes as are necessary to show the variation of the Scheybl version from the Dresden and Vienna codices and from the Arabic manuscript used by Rosen. Professor Karpinski has wisely refrained from attempting a literal translation, since the transcribed text furnishes all needed material for the study of exact expressions; but he has given his readers that free type of translation which permits of a work being easily read and easily understood. All difficulties of any moment are removed by the extensive array of footnotes, and altogether there is little which one could desire that has not been given.

In one sense the Robert of Chester version is not as satisfactory as the one attributed to Gherardo of Cremona, but the latter had already been published by Libri, and hence it was desirable that the former should also be made available for study. In this connection it is also proper to mention another manuscript of Al-Khowarizmi's algebra, for a knowledge of which many students are indebted to Professor Karpinski. This version lay unnoticed in an Italian manuscript of the fifteenth century until the present reviewer happened upon it in the library of the well-known bibliophile George A. Plimpton, of New York, some years ago. This may possibly be the version of William of Luna, and some scholar should do for it what Professor Karpinski has done for the Robert of Chester translation.

Of the work of Al-Khowarizmi itself, this is not the place to speak, since we are concerned with the translation rather than the original. One problem concerning it may, however, be mentioned, namely, that which relates to the source of Al-Khowarizmi's knowledge. Essentially, the treatment is Greek, but no direct connection exists between it and such classical works as those of Euclid and Diophantus. Neither the problems nor the identical methods can be traced to any other source, although Al-Khowarizmi knew something of Hindu mathematics and the Greek authors were already becoming known in Bagdad where he was at work. The problem, therefore, is to determine whether any of this material is to be found in the works of minor Greek or Hindu writers, or possibly in the works of Persian and Chinese authors whose treatises have still to be critically examined.

The work closes with a carefully selected Latin glossary which will be helpful to students of mathematics of the medieval and renaissance periods. It is probably too much to expect that in any university series of this kind the reader is to be assisted by an index. Whether this is because of tradition or because of the economy of our universities it is hard to say; but we may be sure that on this occasion it is not due to the wishes of a student like Professor Karpinski. No one who, like the writer, has had occasion to refer to this work several times, and wishes to find such an item as Gherardo's supposed translation, can fail to regret the omission of this feature.

On the whole, it may be said that the work under review is

a very noteworthy contribution to the study of sources in the history of mathematics.

DAVID EUGENE SMITH.

A First School Calculus. By R. WYKE BAYLISS. London, Longmans, Green and Company, 1915. xii + 288 pp.

THE pedagogical method used in this book is distinctly different from any found in the usual elementary calculus text. The author, a mathematical master at a boys' preparatory school in England, aims to teach the calculus to the youths by means of the question and answer method. Simple and definite questions on concrete problems concerning matter supposedly familiar to the youthful students are used to develop and fix the fundamental principles of the calculus. There are 180 pages of questions and suggestions; the answers to these cover 100 pages.

An equivalent of a meager high school course in mathematics seems sufficient as a prerequisite. Much of the work could be done orally; a private student might make considerable headway by using the text. Graphical work is minimized and included almost entirely among the answers.

No attempt is made to introduce rigor in the derivation of formulas. For example, the formulas based on the exponential function are developed from a practical consideration of the rate of increase of a sum of money placed at compound interest (continuous)—a concept with which all the students are supposed to be familiar. Or they are advised to draw a figure and use this to derive a formula. Or tables of trigonometric functions may be used to get average rates of increase and thus lead to general formulas. All of which, thoroughly rough and ready, seems like substituting a butcher's cleaver with a fairly dull edge for the scalpel in a surgical operation.

In the integral calculus much time and labor is saved by the following definition of integral: "We have seen that the symbol $D^{-1}f(x)$ denotes the expression for the *amount* of a quantity when its rate of increase is denoted by $f(x)$. The *amount* $D^{-1}f(x)$ is called the **integral** of the function $f(x)$." After which formulas may be applied in large chunks. And there is an everlasting amount of formal differentiating and integrating to be done.

The evaluation of the definite integral is arrived at through

the summation process common to most calculus texts. Much smoother sailing is evident when areas, volumes, centroids and moments of inertia are found.

Needless to say, the law of the mean, extended law of the mean, various forms of remainders and their like are decidedly not included in the chapter on the expansion in series, nor are the various degrees of convergence considered. As the author puts it ". . . there are 'pinnacles' and 'caverns' which only the experienced mathematician should explore."

The concluding chapter is headed The Borderland of Discovery. In other texts this is labeled Approximate Integration.

There is throughout the book much material of a rough and ready nature, which is well worth while and of great service in illustrating the fundamental principles of the calculus; there is certainly lacking the close reasoning which only the use of the method of limits can assure the calculus. The question and answer method might work well with a very limited number of students, though they would certainly have to be English because all units used are intensely British; but, with all the answers given in detail, we doubt very much if even the most conscientious students might not too often be tempted to "look in the book and see."

ERNEST W. PONZER.

Mathematische Abhandlungen, Hermann Amandus Schwarz zu seinem fünfzigjährigen Doktorjubiläum am 6. August 1914 gewidmet von Freunden und Schülern. Berlin, Springer, 1914. Portrait, viii+451 pp.

THIS volume forms an imposing testimonial to the influence on the development of modern mathematics of the research and teaching of Schwarz. Of the thirty-four papers contributed by his friends and former students, a majority deal with subjects and methods brought out by him. It is impossible, within the limited space of this review, to do full justice to the rich contents of this volume, so that the reviewer must confine himself to mentioning a few of the papers which have been of particular interest to him, while regretting the necessity of passing in silence many noteworthy contributions.

C. Carathéodory gives a simplification of his former proof in the *Mathematische Annalen*, volume 72, of the most general existence theorem in conformal representation, and establishes

a number of cases in which the continuous one-to-one correspondence of the boundary points may be shown by Schwarz's principle of reflexion. L. Fejér proves some simple and elegant theorems on the convergence of power series, defining a conformal representation, on their convergence circle, and O. Hölder deals with the question of the variation of the solution of a differential equation when the form of the latter is varied. A. Hurwitz solves the problem, proposed by Weierstrass, of the possibility of defining the elliptic sigma function by its addition theorem, while P. Koebe has an article, also inspired by Weierstrass, on analytic functions possessing an algebraic addition theorem. E. Landau carries his researches on prime numbers into definite quadratic forms and pure cubic number fields, Ch. Müntz gives an elegant extension of Weierstrass's theorem on approximation by ordinary polynomials to such as involve non-integral powers, and E. Schmidt presents simple proofs of the fundamental properties of the Newtonian potential, beautiful by their unity of method. J. Schur investigates the expansion of a function in a series of characteristic functions of a positive definite kernel, M. Simon contributes an attractive sketch of the life and works of Sophie Germain, O. Toeplitz gives an example throwing much light upon the scope of Mercer's theorem in integral equations, and in the final paper of the volume, D. Hilbert deals with a general question in the theory of invariants, closely connected with his work in the early nineties.

T. H. GRONWALL.

Graphische Methoden. Von C. RUNGE. Leipzig, Teubner, 1915. iv+142 pp.

THIS book, which appears as No. 18 of Jahnke's collection of mathematical and physical texts, is a translation of the lectures delivered by the author in 1909–10 at Columbia University and published in 1912 as No. 4 of the publications of the Ernest Kempton Adams Fund for Physical Research.

Chapter I gives the means for performing graphically the four elementary operations on real numbers, the graphical calculation of polynomials in one variable and of linear functions of n variables, including the solution of a system of linear equations, and ends with the representation of complex numbers in the Gaussian plane.

Chapter II deals with graphs of functions of one variable,

the principle of the slide rule, change of variables, the calculation of $z = f(x, y)$ by contour lines of the corresponding surface, and its dual method in line coordinates, the nomography of d'Ocagne. The extension of the latter to more than three variables is briefly indicated. Chapter III contains various methods of graphical integration and differentiation, including the determination of the integral curves ot differential equations of the first and second order.

The presentation is concise and very clear, and supported by well chosen illustrative examples and 94 figures, the neatness of which forms a much-needed object lesson to many writers of texts on geometry and graphics.

Regarding literature, there is only a general reference to the corresponding articles in the Encyklopädie; it would have been appropriate to give at least some references for further study, as for instance to d'Ocagne's Calcul graphique et Nomographie, and various papers by Runge, Kutta and others on the graphical integration of differential equations. The book under review brings forth one sad reflection: when will our writers of calculus texts for engineering students see fit to give something really modern and practical on graphical integration and solution of differential equations?

T. H. GRONWALL.

Über die Theorie des Kreisels. Von F. KLEIN und A. SOMMERFELD. Heft I: *Die kinematischen und kinetischen Grundlagen der Theorie.* Zweiter durchgesehener Abdruck. Leipzig, Teubner, 1914. viii+196 pp.

THE second edition of the first part of this standard work differs but slightly from the first one. Literature references have been brought up to date, and occasionally the wording of a theorem is changed.

T. H. GRONWALL.

Konstruktionen in Begrenzter Ebene. Mathematische Bibliothek, herausgegeben von W. LIETZMANN und A. WITTING, XI. Von P. ZÜHLKE. Leipzig und Berlin, B. G. Teubner, 1913. 39 pp. 65 fig.

THIS book treats the subject of constructions in a limited plane primarily from the standpoint of drawing. No restriction is made to a particular set of axioms for proofs, or to any particular set of instruments for constructions. Both metric

and non-metric methods are used, the theorems of Desargues and of Pascal (Pappus) being used for a basis for many of the constructions. Cases in which lines are parallel are considered as distinct from cases in which the lines merely fail to meet on the paper, a distinction which is of special importance in case an instrument for drawing parallels is available.

After a brief historical introduction, the book consists of five subdivisions. I. Unreachable points of intersection of two or more lines. II. Bisection of an angle with non-intersecting sides. III. Construction of triangles and polygons with unreachable vertices. IV. Problems in the theory of circles. V. Bibliography.

A large variety of methods is given, although some of those given as distinct differ in only slight particulars. The chapter on circles, in which points are given by means of non-intersecting circles, i. e., by circles of which parts lie on the paper, but whose points of intersection do not, is especially interesting.

It is much to be regretted that material on such subjects as this is not more readily available in the English language. Perhaps our poverty in well-written elementary books of such a character as to supplement our preparatory-school work is responsible for part of the difficulty in stimulating bright pupils to do work outside of the daily minimum requirement of the textbook. Much of the matter in this little book might well be used for this purpose.

F. W. OWENS.

Introduction géométrique à quelques Théories physiques. Par ÉMILE BOREL. Paris, Gauthier-Villars, 1914. viii+139 pp.

THIS book is centered about the theories of relativity and statistical mechanics, and is divided into two distinct parts, of which the first deals in textbook fashion with certain kinematical questions from a purely mathematical point of view, while the second is composed of seven papers, all published before and rather loosely connected with each other, dealing with various topics in mathematical physics in a critical and philosophical manner. The first part contains four chapters on the euclidean displacements in two and three dimensions, the four-dimensional euclidean geometry, a two-dimensional hyperbolic geometry, and the three- and four-dimensional hyperbolic displacements and their application to the kine-

matics of relativity. A fifth chapter, on functions of a very large number of variables, and areas and volumes in a geometry of 10^{24} dimensions, leads up to statistical mechanics, the number stated being of the order of magnitude of the number of molecules in the unit volume, or the number of dimensions of their velocity space.

The titles of the seven papers forming the second part are as follows: On the principles of the kinetic gas theory; statistical mechanics and irreversibility; the relativity of space according to Henri Poincaré; some remarks on the theory of resonators; on a problem in geometric probability; the kinematics of the theory of relativity; molecular theories and mathematics.

These investigations of some of the most modern questions in theoretical physics should prove of great interest to both mathematicians and physicists.

T. H. GRONWALL.

Grundzüge der Geodäsie. Von M. NÄBAUER. Leipzig, Teubner, 1915. xiv+420 pp.

THIS book forms volume 3 of Handbuch der angewandten Mathematik, edited by H. E. Timerding, and is written primarily with the purpose of acquainting students of mathematics with the modern methods of geodesy. This purpose is quite successfully accomplished by presenting just enough of the practical side of the subject to give the proper setting for the clear and terse mathematical discussion of the underlying principles and the sources of error in the various geodetic operations.

The first part contains the theory of errors and the application of the method of least squares to the reduction of observations. Part two, plane surveying, deals with the surveying instruments, the various kinds of field work (the paragraph on photogrammetry is especially well done), plotting and computation of areas. Part three, higher geodesy, begins with triangulation and the various kinds of coordinates on the earth considered as a sphere, proceeds to the earth ellipsoid, its conformal representation on the sphere and the determination of its dimensions, and ends with a brief account of the determination of the exact figure of the earth by astronomical and pendulum observations.

The mathematical apparatus is confined to the elements of the calculus, and the volume contains much that could be used

to advantage in bringing a course in trigonometry in closer touch with one of its main applications.

T. H. GRONWALL.

Relativity. By A. W. CONWAY. Edinburgh Mathematical Tracts, number 3. London, G. Bell and Sons, 1915. 2 s. 43 pp.

THIS tract is a course of four lectures delivered before the Edinburgh Mathematical Colloquium on the subject of relativity. The audience were representative of various branches of science. These four lectures start with fundamentals, followed by a study of the transformation of the electromagnetic equations, applications to radiation and electron theory, and Minkowski's transformation. The lecturer has succeeded very well in presenting the essentials of the relativity hypothesis free from metaphysics, and speculations of any kind. He has a decidedly sane treatment. There are examples enough to make the ideas clear, stated in everyday terms, and not in terms of the usual mathematical model. It is a serviceable introduction.

JAMES BYRNIE SHAW.

A Theory of Time and Space. By ALFRED A. ROBB. Cambridge, University Press, 1914. vi + 373 pp.

THIS treatise is an elaboration of a previous publication on the same subject. In brief it is an analysis of space and time relations by means of a single type of order called conical. The author also calls the result optical geometry. The treatment is of an axiomatic character, the few diagrams serving only as schemes. There are twenty-one postulates set down, and from these and various definitions, some two hundred and six theorems are deduced. These ultimately lead to statements which permit an algebraic formulation by the use of four parameters, which may be interpreted as the usual x, y, z, and t, the last having a somewhat different rôle from the others. The notion of relativity of course hovers in the background, but any one seeking light on that notion here will be disappointed, as the book is simply a development of a very abstract geometry of four dimensions.

It is not possible to give a resumé of the contents in a review, but some idea can be gained of the point of view by stating

the fundamental conceptions of the author in a somewhat different and more picturesque manner than he does. The model thus given is of course due to the reviewer and not to the author. Let us surround every point of ordinary three-dimensional space with a sphere, the radii being anything we like, and some radii being considered to be positive, some negative. Then any such sphere with its center may be called an element. We will consider that if all the radii are increased by the same amount the elements are unchanged. Two different spheres about the same center at one time are two distinct elements. It is in this conception that the term conical order becomes appropriate, for if we start with a two-dimensional space and surround the points by circles, when the radii are increased by the same amount we may move the plane of operations parallel to itself a proportional amount, thus generating cones by the expanding circles. This conception however cannot be used in the imagery of three dimensions.

The element B is said to be *after* the element A when the distance of the center of A from that of B is less than the algebraic difference of the radii of B and A, this difference being positive. If the distance between the centers is less than the difference between the radii of A and B, this difference being positive, B is said to be *before* A. If the distance between centers is greater than the absolute value of the difference of the radii neither A nor B is before or after the other. This is the type of order upon which the author bases his whole development. The appropriateness of the terms from an optical point of view is seen if we consider the spheres to be the wave-fronts of light signals from the points. For a signal to proceed from A to B within a given time, the element B must be after the element A.

The α-subset of A consists of all elements whose centers and radii are such that the distances from the center of A to the other centers are equal to the differences between the radius of A and the radii of the other elements. The β-subset of A consists of all elements whose distances from the center of A equal the differences of their radii and the radius of A. The element A belongs to both subsets. If the radii are all increased sufficiently the α-subset will consist of all elements whose spheres are internally tangent to that of A, or would become so by further increase of radii.

The elements A_1 and A_2 determine an *optical line* if A_2 is an element of either the α-subset of A_1 or its β-subset. The optical line then consists of all elements which are in either the two α-subsets of A_1 and A_2 or their two β-subsets. All these elements will be tangent to the sphere of A_1 at the same point, or will become so when the radii are all sufficiently increased. If A_2 is neither before nor after A_1, then the two determine a *separation line*, because the elements which have centers on this line may have spheres of such size that no one is before or after any other. In this sense they are like particles on a line in space. If A_2 is not in the α-subset of A_1 but is after A_1 they determine an *inertia line*, in the sense that a particle properly chosen could move along this line with a given velocity, that is to say elements may be so chosen as to be successively after A_1 and before A_2. A separation segment is said to have a length r, which is the second side of a right triangle made on the distance between centers as hypotenuse, and the difference of the radii as first side. An inertia line is said to have a length which is the second side of a right triangle whose hypotenuse is the difference of radii and first side the distance between centers. In the latter case however the unit of measure is taken to be a standard number v called the velocity of light, and which would be the rate of increase of all the radii of all the elements.

It is clear that the time deduced in such an axiomatic treatment is not the Time of philosophy nor of psychology, but is merely a kind of order. The time involved here does not flow, for the particular system of spheres we chose is a stationary set, and any increase in radii is for convenience merely, since only the differences of the radii are ever considered. Time enters only by constructing a second system of spheres about all the points of space. The new radii are then representative of times (instants) different from those represented by the first set. As the author claims, he is studying a type of order, which permits of the abandonment of the notion of simultaneity save as a local phenomenon. In this sense he is studying relativity. The treatise will be interesting in the main to students of postulational geometries.

JAMES BYRNIE SHAW.

Repertorium der Physik. Von R. H. WEBER und R. GANS. Erster Band: *Mechanik und Warme.* Erster Teil. Leipzig, B. G. Teubner, 1915. xii + 434 pp. 8vo. Price, 8 Marks.

THIS is the first part of the first volume of a repertorium of physics similar to the well-known Pascal repertorium of mathematics. The general character is more that of a small treatise, intermediate between an ordinary textbook and a manual, since it is written in a fairly connected style. It includes many results of memoirs, however, that one would not find in the ordinary text, and omits many details, particularly numerical ones, that would be found in a complete manual. In this way it has been possible to compress within a reasonable space a quite convenient vademecum for both physicists and mathematicians.

This first part is divided into three books. The first and second are written by R. Gans of La Plata. The third book is written by F. A. Schulze. The first book is on the mechanics of discrete particles. A list of the chapter headings will convey a sufficient idea of the contents. These are: Fundamentals of mechanics; Principles of mechanics; Dynamics of rigid bodies; Gravitation; Coordinate systems, rotation of the earth, centrifugal force; Friction; Vibrations. The second book is devoted to the mechanics of continuous media under two divisions: *A*, Elasticity; *B*, Hydrodynamics. Under *A* we find chapters on: Kinematics and dynamics of deformable media; Statical problems of the theory of elasticity; Dynamical problems of the theory of elasticity. Under *B* we find the chapters treating of: Equations of motion and general theorems; Kinematics of fluids with an immersed fixed body; Dynamics of fluids containing an immersed fixed body; Problems in two dimensions; Waves; Vibrations of the air; Viscosity; Tides. The book on Acoustics contains chapters on: Propagation of sound; Intensity of sound; Various problems; Musical scales.

The second part of the first volume will treat of Capillarity, Heat, Statistical mechanics, Kinetic theory of gases; the second volume will be devoted to Electricity, Magnetism, and Optics.

It may be regretted that for a subject as extensive as physics, the work is not a little more comprehensive, but as it is it will be useful.

JAMES BYRNIE SHAW.

NOTES.

THE April number of the *Transactions of the American Mathematical Society* contains the following papers: "On multiform solutions of linear differential equations having elliptic function coefficients," by W. L. MISER; "On the foundations of plane analysis situs," by R. L. Moore; "On the generalized Jacobi-Kummer cyclotomic function," by H. H. MITCHELL; "Proof of a theorem of Haskins," by D. JACKSON; "On the measurable bounds and the distribution of functional values of summable functions," by C. N. HASKINS; "Jacobi's condition for problems of the calculus of variations in parametric form," by G. A. BLISS.

THE twenty-first summer meeting and eighth colloquium of the American Mathematical Society will be held at Harvard University during the week beginning Monday, September 4, 1916. The first two days will be devoted to the regular sessions for the presentation of papers. The colloquium will open on Wednesday morning and close on Saturday morning. Two courses of five lectures each will be given as follows (the list of principal topics is appended):

PROFESSOR G. C. EVANS: "Topics from the theory and applications of functionals, including integral equations."

The lectures will attempt a brief survey of the present state of this theory, showing how its results have been applied in various branches of analysis and in physics. The following topics will be considered, with the purpose of keeping the point of view as general as possible:

Functions depending on curves and surfaces in three dimensions, determination of Green's function by means of variational equations; functions depending on curves and surfaces in four dimensions, integrals of analytic functions of two complex variables; the linear functional relation, the integral equation of the third kind; applications of the Volterra theory of functional relations; applications of the Hilbert theory of integral equations.

PROFESSOR OSWALD VEBLEN: "Analysis situs."

This course will attempt to give an account of the present state of this elementary but relatively undeveloped branch of geometry. Among the topics considered will be:

The n-dimensional cell; separation of a cell into regions by polyhedra; combinatorial properties of polyhedra; manifolds as generalized polyhedra; abstract equivalence of manifolds; numerical invariants and group of a manifold; equivalence of two manifolds within a third, theory of knots; continuous transformations of a manifold into itself.

The chief references can be obtained from the article on Analysis Situs in the Encyklopädie and the chapter on Topology in the new edition of Pascal's Repertorium.

AT the meeting of the London mathematical society held March 9 the following papers were read: By P. A. MACMAHON: "Some applications of general theorems of combinatory analysis"; by H. F. BAKER: "Mr. Grace's theorem on six lines with a common transversal"; by H. E. J. CURZON: "The integrals of a certain Riccati equation connected with Halphen's transformation"; by HILDA P. HUDSON: "A certain plane sextic"; by W. P. MILNE: "The construction of co-apolar triads on a cubic curve"; by J. BONDMAN: "The dynamical equations of the tides."

AT the meeting of the Edinburgh mathematical society on March 10 the following papers were read: By J. F. TINTO: "Transformations founded on the space cubic and its chord system"; by J. DOUGALL: "Elliptic cylindrical harmonics."

THE Association of mathematics teachers of New Jersey held its fourth regular meeting at Princeton University on May 6, 1916. The programme included the presidential address of Professor H. B. FINE: "The theory of incommensurable magnitudes as set forth in the tenth book of Euclid's Elements;" Report of the committee on trigonometry courses; EDWIN FLORANCE: "Ptolemy's theorem;" E. S. INGHAM: "An exposition of Napier's principle of logarithms;" Rev. F. C. DOAN: "Certain religious implications of the mathematical infinite;" J. C. STONE: "The ultimate aim of a course in arithmetic."

THE Paris academy of sciences announces as the subject for its Grand prize in mathematics (3000 francs) for 1917 the following:

"To perfect in an important point the theory of successive

powers of a given substitution, the exponent of the power increasing indefinitely. The influence of the initial element is to be considered, and the consideration may be limited to the simplest cases, such as that of rational substitutions in one variable."

The Academy has elected as corresponding members Professors LIAPOUNOF, of Petrograd, and C. J. DE LA VALLEÉ POUSSIN, of Louvain.

ON account of the war two mathematical periodicals have suspended publication, viz., *L'Education Mathématique*, which concluded its sixteenth and last volume with the issue for July, 1914, and the *Revue de Mathématiques Spéciales*, last issued in September, 1914, in completion of its twenty-fourth consecutive year.

THE following university courses in mathematics are announced for the academic year 1916–1917:

CORNELL UNIVERSITY.—By Professor J. MCMAHON: Theory of probabilities, three hours.—By Professor V. SNYDER: Descriptive geometry (first term), three hours; Algebraic curves and surfaces, three hours.—By Professor F. R. SHARPE: Theory of potential and Fourier series, three hours.—By Professor W. B. CARVER: Differential geometry (first term), three hours; Theory of numbers (second term), three hours.—By Professor A. RANUM: Modern algebra (second term), three hours.—By Professor D. C. GILLESPIE: Principles of mechanics, three hours.—By Professor W. A. HURWITZ: Theory of functions of real variables, three hours.—By Professor C. F. CRAIG: Functions of a complex variable, three hours.—By Professor F. W. OWENS: Differential equations, three hours; Mathematical physics, three hours.—By Dr. J. V. MCKELVEY: Advanced calculus, three hours.—By Dr. L. L. SILVERMAN: Analytic geometry, three hours.—By Dr. M. G. GABA: Projective geometry, three hours.—By Mr. H. BETZ: Graphical processes and numerical calculation, three hours.

HARVARD UNIVERSITY.—All courses meet three times a week throughout the year, except those marked*, which meet for half a year.—By Professor W. F. OSGOOD: Introduction to potential functions and Laplace's equation;* Galois's theory

of equations.*—By Professor M. BÔCHER: Interpolation and approximation;* Theory of functions; Linear differential equations, complex variables.*—By Professor C. L. BOUTON: Advanced calculus; Geometrical transformations, with special reference to the work of Sophus Lie.—By Professor J. L. COOLIDGE: Modern geometry and modern algebra; Line geometry.—By Professor E. V. HUNTINGTON: Fundamental concepts of mathematics.*—By Professor G. D. BIRKHOFF: Dynamics, second course;* Analytical theory of heat, Fourier's series, and Legendre's polynomials;* Applications of the calculus of variations.*—By Professor D. JACKSON: Infinite series and products;* Lebesgue integrals.*—By Dr. G. M. GREEN: Elementary differential equations;* Differential geometry of curves and surfaces.—By Dr. E. KIRCHER: Vector analysis;* Finite groups.*—By Mr. W. LeR. HART: Calculus of variations;* Functions of infinitely many variables.*

Professors Osgood and Birkhoff will conduct a fortnightly seminar in analysis.

Courses of research are also offered by Professor Osgood in the theory of functions, by Professor Bôcher in the real solutions of linear differential equations, by Professor Bouton in the theory of point-transformations, by Professor Coolidge in geometry, by Professor Birkhoff in the theory of differential equations, by Professor Jackson in the theory of functions of a real variable, and by Dr. Green in differential geometry.

UNIVERSITY OF PENNSYLVANIA.—By Professor E. S. CRAWLEY: Modern analytic geometry.—By Professor G. E. FISHER: Theory of functions of a complex variable.—By Professor G. H. HALLETT: Galois theory of equations.—By Professor F. H. SAFFORD: Partial differential equations.—By Professor M. J. BABB: Introduction to modern higher algebra.—By Professor O. E. GLENN: Theory of invariants.—By Professor H. H. MITCHELL: Elliptic functions.—By Dr. R. L. MOORE: Functions of a real variable with an introduction to certain phases of general analysis.—By Dr. F. W. BEAL: Differential geometry.

PRINCETON UNIVERSITY (1916-1917).—By Professor H. B. FINE: History of analysis, second term, three hours.—By Professor L. P. EISENHART: Differential geometry, three hours.—By Professor OSWALD VEBLEN: Projective geometry,

three hours; Seminar.—By Professor PIERRE BOUTROUX: Analysis, three hours; Differential equations, three hours.—By Dr. A. A. BENNETT: Algebra, three hours; Projective geometry, three hours.—By Dr. J. W. ALEXANDER: Newtonian potential function, three hours.

SHELDON travelling fellowships for the year 1916–1917 have been awarded by Harvard University to Mr. R. W. BRINK and Mr. J. L. WALSH. During the current academic year a similar fellowship is held by Mr. L. R. FORD, who is studying in Paris. Mr. A. L. MILLER holds a Rogers travelling fellowship and is studying in Turin.

ONE of the two fellowships, of the annual value of seven hundred and fifty dollars, in the department of mathematics of the Rice Institute remains to be filled for the academic year 1916–1917. The successful candidate will be expected to enter upon a course of study and research work leading to the degree of doctor of philosophy, and also to assist with elementary teaching of mathematics for six hours per week. The fellows will be able to live in the residential hall of the Institute, where board and lodging will be provided for them at about thirty dollars per month. Applications accompanied by testimonials and a full statement of previous work and training should be addressed to the Department of Mathematics, Rice Institute, Houston, Texas.

ON March 16, 1916, the Scandinavian mathematicians celebrated at Stockholm the seventieth birthday of their illustrious colleague, Professor G. MITTAG-LEFFLER, founder and director of the *Acta Mathematica*. Many messages of congratulation were received from mathematicians of other countries. On this occasion Professor Mittag-Leffler and his wife set aside their entire fortune for the foundation of an International institute for pure mathematics.

MR. R. E. GILLMAN, of Princeton University, has been appointed instructor in mathematics at Cornell University.

NEW PUBLICATIONS.

I. HIGHER MATHEMATICS.

BACKES (W.). Ein Nachtrag zu den Beweisen für den Fermatschen Satz $x^n + y^n = z^n$. Mainz, 1915. 4 pp. M. 3.00

BELL (E. T.). Arithmetical theory of certain numerical functions. (University of Washington Publications in Mathematical and Physical Sciences, vol. 1, No. 1.) Seattle, University of Washington, 1915. 4to. 44 pp. $0.50

BÔCHER (M.). Syllabus of a brief course in solid analytic geometry. (Rectangular coordinates.) Cambridge, Mass., Harvard Cooperative Society, 1916. 8vo. 10 pp. Paper. $0.15

BONYNGE (W.). See FISHER (A.).

FISHER (A.). The mathematical theory of probabilities and its application to frequency curves and statistical methods. Translated and edited from the author's original Danish notes with the assistance of W. Bonynge, with an introductory note by F. W. Frankland. Vol. 1. Mathematical probabilities and homograde statistics. New York, Macmillan, 1915. 20+171 pp. $2.00

FORD (W. B.). Studies on divergent series and summability. (University of Michigan Studies. Scientific Series, vol. 2.) New York, Macmillan, 1916. 4to. 12+194 pp. Cloth. $2.50

FRANKLAND (F. W.). See FISHER (A.).

FUBINI (G.). Lezioni di analisi matematica. 2a edizione, interamente rifusa. (Grande biblioteca tecnica, no. 9.) Torino, Soc. tip. ed. Nazionale, 1915. 8vo. L. 14.00

LEWIS (F. P.). Geometrical application of the theory of the binary quintic. (Diss.) Baltimore, Johns Hopkins, 1914. 4to. 24 pp.

RULF (W.). Analytische Geometrie für höhere Gewerbeschulen. Wien, Deuticke, 1915. 158 pp. Geb. Kr. 3.20

SPOLTORE (N.). La duplicazione grafica del cubo. Vasto, tip. Zaccagnini, 1915. 8vo. 8 pp. con tavola.

UDZIELA (E.). Neue Lösung des Fermatschen Problems $x^n + y^n = z^n$. Wien, Udziela, 1915. 6 pp.

II. ELEMENTARY MATHEMATICS.

ANDERSON (R. F.). See PHILIPS (G. M.).

BEHRENDSEN (O.) und GÖTTING (E.). Lehrbuch der Mathematik nach modernen Grundsätzen. Leipzig, Teubner, 1915. 8vo. Ausgabe B für Schulen mit realistischem Lehrplan. Unterstufe. 3te Auflage. 8+350 pp. Oberstufe. 2te Auflage. 7+404 pp. M. 2.80+4.00

BENEDICT (H. Y.) and CALHOUN (J. W.). Teaching of plane geometry. 2d issue. (Official series, No. 104.) Austin, University of Texas, 1914. 8vo. 57 pp.

BETZ (W.) and WEBB (H. E.). Solid geometry. With the editorial co-operation of P. F. Smith. Boston, Ginn, 1916. Pp. 22+327–504. $0.75

BROOKMAN (T. A.). A practical algebra for beginners. New York, Scribner, 1915. 8vo. 17+322 pp. $1.12

CALHOUN (J. W.). See BENEDICT (H. Y.).

COURTIS (S. A.). Courtis standard practice tests in the four operations with whole numbers. (Efficiency drill series.) Three parts. Yonkers-on-Hudson, World Book Company, 1914. Teacher's manual, paper, $0.50. Practice tests (48 lesson leaves), $0.16. Student's record, per package of 50, $1.00.

FISCHER (P. B.). Rechenbuch für die unteren und mittleren Klassen der höheren Knabenschulen. Leipzig, Teubner, 1915. Ausgabe in 1 Bande. 4+196 pp. M. 2.20. Ausgabe in 3 Heften. 1ter Teil: Lehrstoff der Sexta. 4+73 pp. M. 1.00. 2ter Teil: Lehrstoff der Quinta. 4+62 pp. M. 0.80. 3ter Teil: Lehrstoff der Quarta. 4+60 pp. M. 0.80.

GIFFORD (J. B.). Everyday arithmetic. Boston, Little, Brown and Company, 1916. $0.35

GÖTTING (E.). See BEHRENDSEN (O.).

HAMILTON (S.). Key to Hamilton's Arithmetics, second and third books; New Jersey edition. New York, American Book Company, 1915. 188 pp. $0.64

HART (W. W.). See WELLS (W.).

HARTL (H.). Lehrbuch der Planimetrie. 3te Auflage. Wien, Deuticke, 1915. 143 pp. Geb. Kr. 3.00

HARVEY (L. D.). Essentials of arithmetic; with everyday problems relating to agriculture, commerce, and other vocations. (Three book series.) 2d book, parts 1–2. New York, American Book Company, 1915. $0.36+0.36

HOLLOWAY (H. V.). Experimental study to determine the relative difficulty of the elementary number combinations in addition and multiplication. Bordentown, N. J., Holloway, 1915. 8vo. 102 pp. $1.00

HOYT (F. S.) and PEET (H. E.). Everyday arithmetic. 3 books. Boston, Houghton-Mifflin, 1915. 138 +140+190 pp. $0.40+0.40+0.45

JACOB (J.). Arithmetik. 3ter Teil: Oberstufe. 2te Auflage. Wien, Deuticke, 1915. 111 pp. Schlüssel dazu. 84 pp. Geb. Kr. 2.20+2.40

JESSUP (W. A.). Economy of time in arithmetic. (University of Iowa extension bulletin, No. 5.) 8vo. Pp. 461–476. Paper. Gratis.

MILLIS (J. F.). See STONE (J. C.).

MILNE (W. J.). Key to New York state arithmetic; first and second books. New York, American Book Company, 1915. 304 pp. $0.80

MINER (G. W.). See MOORE (J. H.).

MLODZIEVSKY (B.). Rapport sur l'enseignement mathématique aux cours supérieurs des femmes à Moscou. (Commission internationale de l'enseignement mathématique. Sous-commission russe.) Petrograd, Luinik, 1915. 20 pp.

MOORE (J. H.) and MINER (G. W.). Concise business arithmetic. Boston, Ginn, 1915. 8vo. 6+283 pp. $0.75

MUNDT (J.). Rechenaufgaben über den Weltkrieg für die deutsche Jugend. Cöln, P. Schmitz, 1915. 24 pp. M. 0.10

OBERG (E. V.). Elementary algebra. (Machinery's reference series, No. 138.) New York, Industrial Press, 1914. 40 pp. Paper. $0.25

PEET (H. E.). See HOYT (F. S.).

PENDLETON (F. T.). Modern arithmetic. New London, Conn., F. T. Pendleton, 1914. 37 pp. $0.25

PHILIPS (G. M.) and ANDERSON (R. F.). Course of study in arithmetic for grades below the high school. Boston, Silver, Burdett and Company, 1915. 45 pp. Gratis.

RUBINSTEIN (A.). Review in algebra for those who know algebra but want to know it well enough to pass an examination. Revised edition. New York, Hinds, Noble and Eldridge, 1915. 60 pp. $0.25

SMITH (P. F.). See BETZ (W.).

SOMERVILLE (F. H.). Key to elementary algebra. Revised edition. New York, American Book Company, 1915. 374 pp. $1.00

STONE (J. C.) and MILLIS (J. F.). Manual and course of study for teachers of arithmetic. Chicago, Sanborn, 1914. 38 pp. $0.10

WAGNER (E.). Repetitorium der Mathematik. 2ter Teil: Trigonometrie, Stereometrie, Maxima und Minima. Strassburg in Els., Van Hauten, 1914. 92 pp.

WEBB (H. E.). See BETZ (W.).

WEINTROB (R.). Silk arithmetic. New York, Simmons, 1915. 47 pp. $0.35

WELLS (W.) and HART (W. W.). Plane and solid geometry. Boston, Heath, 1916. 8vo. 10+467 pp.

III. APPLIED MATHEMATICS.

ANNUAIRE astronomique et météorologique. Paris, Flammarion, 1916. 431 pp.

ANNUAIRE pour l'an 1916 publié par le Bureau des Longitudes. Paris, Gauthier-Villars, 1915. 6+502 pp. Fr. 1.50

ANNUARIO del observatorio de Madrid para 1916. Madrid, Bailly-Balliere, 1916. 645 pp.

ASHWORTH (J. R.). An introductory course of practical magnetism and electricity. 3d edition. London, Whittaker, 1916. 17+96 pp. 2s.

AUERBACH (F.). Die Physik im Kriege. 2te Auflgae. Jena, Fischer, 1915. 4+209 pp. M. 4.00

BOWLEY (A. L.). Elementary manual of statistics. (Modern commercial text-books.) New York, Scribner, 1914. 8vo. 215 pp. $2.00

DADOURIAN (H. M.). Analytical mechanics. 2d edition. New York, Van Nostrand, 1915. $3.00

ERWIN (M.). The universe and the atom. London, Constable, 1916. 8vo. 8s. 6d.

EVANS (C. T.). Arithmetic of elementary chemistry. Pottstown, Pa., Hill School, 1914. 99 pp. $1.00

FREUCHEN (P. B.). Termodynamik. Kjöbenhavn, Lehmann, Stages Forlag, 1916. 144 pp.

HALER (P. J.) and STUART (A. H.). A first course in engineering science. London, University Tutorial Press, 1915. 8+191 pp. 2s. 6d.

HAWKINS (N.). Self-help mechanical drawing; an educational treatise. New York, Audel, 1915. 8vo. 13+299 pp. $2.00

HUTTON (P. W.). Mechanical drawing for industrial and continuation schools. Chicago, Scott, Foresman and Company, 1915. 8vo. 176 pp. $0.90

KOITCAMP (J. P.). Elementary mechanics for the practical engineer; engineer's study course from power. New York, McGraw-Hill, 1915. 8vo. 7+181 pp. $1.50

LEEDS (C. C.). Mechanical drawing for technical and high schools. 3d edition. New York, Van Nostrand, 1915. $1.25

MAILLOUX (C. O.). See SEMENZA (G.).

MASO (R. P. M. S.). Historia del observatorio de Manila, 1865–1915. Manila, E. C. McCullaugh, 1915. 210 pp.

MEYER (A. F.). Graphs of meteorological and hydrological data. Minneapolis, Northwestern School Supply, 1915. 8vo. 47 pp. Paper. $0.15

MILLER (D. C.). The science of musical sounds. New York, Macmillan, 1916. $2.50

PÜNING (H.). Lehrbuch der Physik. 10te Auflage. Münster, Aschendorff, 1915. 35[illegible] pp. M. 3.70

SCHRAIDT (F. F.). Geometrical drawing; a selection of plates for practical use in elementary mechanical drawing. San Francisco, Whittaker and Ray-Wiggin, 1915. 8vo. 78 pp. $0.65

SEMENZA (G. and M.). Graphical determination of sags and stresses for overhead line construction. Translated from the Italian by C. O. Mailloux. New York, McGraw-Hill, 1915. 10+24 pp. $3.00

SEMENZA (M.). See SEMENZA (G.).

SMITH (E.). Evolution of a gravitating, rotating, condensing fluid. (University of Cincinnati Studies.) Cincinnati, 1915. 45 pp. Paper. $0.50

STUART (A. H.). See HALER (P. J.).

THOMAS (C. M.). Compass surveying and the simplified calculation of farm areas. Wytheville, Pa., C. M. Thomas, 1915. 8vo. 6+92 pp. $2.00

WEICH (C. W.). Elementary mechanical drawing; theory and practice, with chapters on geometrical drawing, mensuration, and reproduction of drawings: a text-book for technical, secondary, trade and vocational schools. New York, McGraw-Hill, 1915. 8vo. 10+250 pp. $1.75